PHYS 7895 Fall 2013
Introduction to Quantum Information Theory
Homework 2

Due Tuesday 15 October 2013, by 5pm in Nicholson 451 (Professor Hwang Lee's office)
(You are allowed to work with others as long as you write down who your collaborators are. Any late assignments will be penalized in the amount of 25% per day late.)

This assignment has a first part and a second part.

First part: Exercises in **arXiv:1106.1445**:

3.5.11, 3.6.6, 3.6.8, 3.6.9, 3.6.12, 4.1.5, 4.1.6, 4.1.7, 4.1.8, 4.1.11, 4.2.1, 4.3.3, 4.3.4

Second part: The following exercises:

1. You will prove that the Schmidt decomposition gives a way to identify if a pure state is entangled or product. In particular, you will prove that a pure bipartite state is entangled if and only if it has more than one Schmidt coefficient. (To understand what we mean by Schmidt coefficient, suppose that a state $|\phi\rangle_{AB}$ has the following Schmidt decomposition:
$$|\phi\rangle_{AB} = \sum_k \lambda_k |\chi_k\rangle_A \otimes |\theta_k\rangle_B,$$
for orthonormal bases $\{|\chi_k\rangle_A\}$ and $\{|\theta_k\rangle_B\}$. The number of Schmidt coefficients is the number of non-zero λ_k in the above sum.)

    (a) First, suppose that a pure bipartite state $|\phi\rangle_{AB}$ has only one Schmidt coefficient. Prove that its maximum overlap with a product state is equal to one:
$$\max_{|\varphi\rangle_A, |\psi\rangle_B} \left|\langle\varphi|_A \otimes \langle\psi|_B |\phi\rangle_{AB}\right|^2 = 1.$$

    (b) Now, suppose that there is more than one Schmidt coefficient for a state $|\phi\rangle_{AB}$. Prove that this state's maximum overlap with a product state is strictly less than one (and thus it cannot be written as a product state):
$$\max_{|\varphi\rangle_A, |\psi\rangle_B} \left|\langle\varphi|_A \otimes \langle\psi|_B |\phi\rangle_{AB}\right|^2 < 1.$$

    (*Hint: Use the Schmidt! Use the Schwarz! (as in Cauchy-Schwarz...)*)

2. Suppose that it is possible for a tripartite state ρ_{ABC} to exist such that $\text{Tr}_B\{\rho_{ABC}\}$ is equal to a maximally entangled state and such that $\text{Tr}_C\{\rho_{ABC}\}$ is equal to a maximally entangled state. Show using the teleportation protocol that it is impossible for such a state to exist, because it would lead to a violation of the no-cloning theorem. (The fact that one system cannot be maximally entangled with two separate parties is known as monogamy of entanglement, or if you're a negative person, you could call it the "no promiscuous entanglements" restriction.)

3. Suppose that you are given a description of a unitary circuit (implementing a unitary U) to make a mixed quantum state ρ on system S, in the sense that if you perform the circuit on n qubits initialized to the state $|0\rangle^{\otimes n}$, you get the state ρ by tracing out some of the qubits in a system R:

$$\rho_S = \mathrm{Tr}_R\{U\big(|0\rangle\langle 0|^{\otimes n}\big)U^\dagger\}.$$

Show that the circuit in Figure 1 can be used to estimate the purity $\mathrm{Tr}\{\rho^2\}$ of the state ρ. (Hint: Use the Schmidt!)

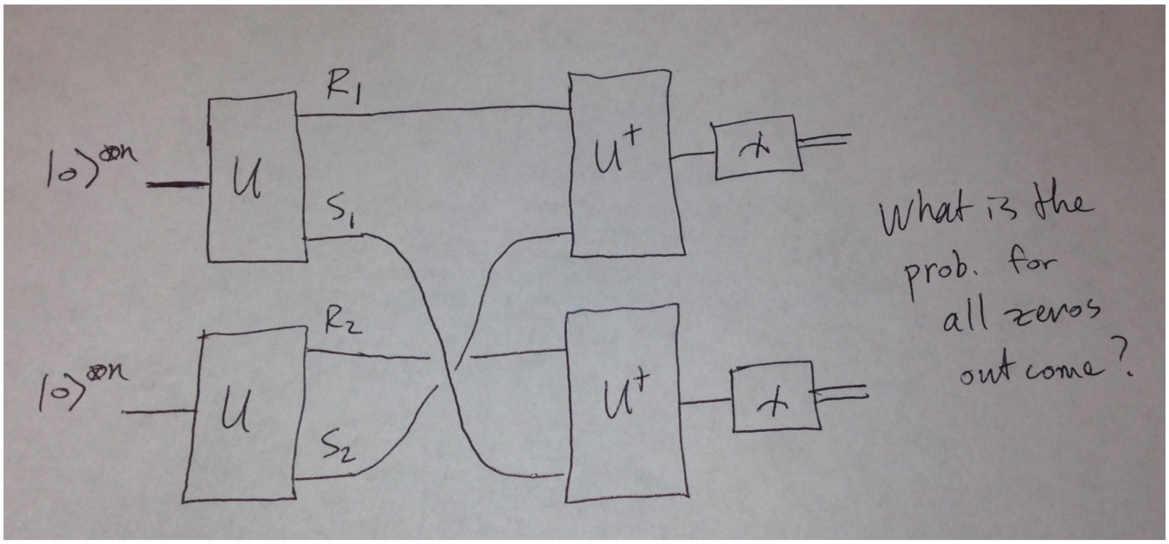

Figure 1: Circuit for estimating the purity of a quantum state.

Bonus: Show how to do this using the so-called SWAP test.