Proof of the Choi-Kraus theorem

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Abstract

We provide a simple proof of the Choi-Kraus theorem (as a reference).

Every evolution of a quantum state should satisfy three properties:

1. It should be linear so that we do not allow for signaling (by the steering argument, one could have signaling).
2. It should be completely positive so that it takes quantum states to quantum states (even for systems correlated with the one on which the map is acting).
3. It should be trace preserving (again so that it takes quantum states to quantum states).

The three requirements above lead naturally to the Choi-Kraus representation theorem, which states that the map has to take a particular form according to a Choi-Kraus decomposition:

Theorem 1 (Choi-Kraus) *A map $\mathcal{N}_{A\to B}$ from a finite-dimensional Hilbert space $\mathcal{H}_A$ to a finite-dimensional Hilbert space $\mathcal{H}_B$ is linear, completely positive, and trace-preserving if and only if it has a Choi-Kraus decomposition as follows:*

$$\mathcal{N}_{A\to B}(X_A) = \sum_{l=0}^{d-1} V_l X_A V_l^\dagger, \tag{1}$$

where $X_A : \mathcal{H}_A \to \mathcal{H}_A$, $V_l : \mathcal{H}_A \to \mathcal{H}_B$ for all $l \in \{0, \ldots, d-1\}$,

$$\sum_{l=0}^{d-1} V_l^\dagger V_l = I_A, \tag{2}$$

and $d \leq \dim(\mathcal{H}_A)\dim(\mathcal{H}_B)$.

Proof. We first prove the easier "if-part" of the theorem. So let us suppose that $\mathcal{N}_{A\to B}$ has the form in (1) and that the condition in (2) holds as well. Then $\mathcal{N}_{A\to B}$ is clearly a linear map. It is

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completely positive because $(\mathrm{id}_R \otimes \mathcal{N}_{A\to B})(X_{RA}) \geq 0$ if $X_{RA} \geq 0$ when $\mathcal{N}_{A\to B}$ has the form in (1), and this holds for a reference system R of arbitrary size. That is, consider that

$$(\mathrm{id}_R \otimes \mathcal{N}_{A\to B})(X_{RA}) = \sum_{l=0}^{d-1} (I_R \otimes V_l) X_{RA} (I_R \otimes V_l^\dagger) \tag{3}$$

$$= \sum_{l=0}^{d-1} (I_R \otimes V_l) X_{RA} (I_R \otimes V_l)^\dagger. \tag{4}$$

We know that $(I_R \otimes V_l) X_{RA} (I_R \otimes V_l)^\dagger \geq 0$ for all l when $X_{RA} \geq 0$, and the same is true for the sum. Trace preservation follows because

$$\mathrm{Tr}\left\{\mathcal{N}_{A\to B}(X_A)\right\} = \mathrm{Tr}\left\{\sum_{l=0}^{d-1} V_l X_A V_l^\dagger\right\} \tag{5}$$

$$= \mathrm{Tr}\left\{\sum_{l=0}^{d-1} V_l^\dagger V_l X_A\right\} \tag{6}$$

$$= \mathrm{Tr}\left\{X_A\right\}, \tag{7}$$

where the second line is from cyclicity of trace and the last line follows from the condition in (2).

We now prove the more difficult "only-if" part. Let $d_A \equiv \dim(\mathcal{H}_A)$ and $d_B \equiv \dim(\mathcal{H}_B)$. Let $|\Gamma\rangle_{RA}$ denote the following unnormalized maximally entangled vector:

$$|\Gamma\rangle_{RA} \equiv \sum_{i=0}^{d_A-1} |i\rangle_R \otimes |i\rangle_A\,, \tag{8}$$

where $\{|i\rangle_A\}$ is an orthonormal for the A system and $\{|i\rangle_R\}$ is an orthonormal basis for an auxiliary R system. The Choi matrix of a completely-positive, trace-preserving (CPTP) linear map $\mathcal{N}_{A\to B}$ is defined as follows:

$$\mathcal{N}_{A\to B}\left(|\Gamma\rangle\langle\Gamma|_{RA}\right) = \sum_{i,j=0}^{d_A-1} |i\rangle\langle j|_R \otimes \mathcal{N}_{A\to B}\left(|i\rangle\langle j|_A\right). \tag{9}$$

This matrix completely describes the action of the map because it describes the action of it on every operator $|i\rangle\langle j|_A$, from which we can construct any other operator on which the map acts, due to the fact that $\mathcal{N}_{A\to B}$ is linear (the Choi matrix is a large $d_A d_B \times d_A d_B$ matrix with blocks of the form $\mathcal{N}_{A\to B}\left(|i\rangle\langle j|_A\right)$). Also, the above matrix is positive semidefinite due to the requirement that the map is completely positive. So we can diagonalize $\mathcal{N}_{A\to B}\left(|\Gamma\rangle\langle\Gamma|_{RA}\right)$ as follows:

$$\mathcal{N}_{A\to B}\left(|\Gamma\rangle\langle\Gamma|_{RA}\right) = \sum_{l=0}^{d-1} |\phi_l\rangle\langle\phi_l|_{RB}\,, \tag{10}$$

where $d \leq d_A d_B$ is the Choi rank of the map $\mathcal{N}_{A\to B}$. (This decomposition does not necessarily have to be such that the vectors $\{|\phi_l\rangle_{RB}\}$ are orthonormal.) Consider by inspecting (9) that

$$\left(\langle i|_R \otimes I_B\right)\left(\mathcal{N}_{A\to B}\left(|\Gamma\rangle\langle\Gamma|_{RA}\right)\right)\left(|j\rangle_R \otimes I_B\right) = \mathcal{N}_{A\to B}\left(|i\rangle\langle j|\right). \tag{11}$$

Now, consider that for any bipartite vector $|\phi\rangle_{RB}$, we can expand it in terms of an orthonormal basis $\{|j\rangle_B\}$ and the basis $\{|i\rangle_R\}$ given above:

$$|\phi\rangle_{RB} = \sum_{i=0}^{d_A-1}\sum_{j=0}^{d_B-1} \alpha_{ij} |i\rangle_R \otimes |j\rangle_B . \tag{12}$$

Let $V_{A\to B}$ denote the following linear operator:

$$V_{A\to B} \equiv \sum_{i=0}^{d_A-1}\sum_{j=0}^{d_B-1} \alpha_{i,j} |j\rangle_B \langle i|_A , \tag{13}$$

where $\{|i\rangle_A\}$ is the orthonormal basis given above. Then we see that

$$\begin{aligned}(I_R \otimes V_{A\to B}) |\Gamma\rangle_{RA} &= \sum_{i=0}^{d_A-1}\sum_{j=0}^{d_B-1} \alpha_{i,j} |j\rangle_B \langle i|_A \sum_{k=0}^{d_A-1} |k\rangle_R \otimes |k\rangle_A &(14)\\ &= \sum_{i=0}^{d_A-1}\sum_{j=0}^{d_B-1}\sum_{k=0}^{d_A-1} \alpha_{i,j} |k\rangle_R \otimes |j\rangle_B \langle i|k\rangle_A &(15)\\ &= \sum_{i=0}^{d_A-1}\sum_{j=0}^{d_B-1} \alpha_{ij} |i\rangle_R \otimes |j\rangle_B &(16)\\ &= |\phi\rangle_{RB} . &(17)\end{aligned}$$

So this means that for all bipartite vectors $|\phi\rangle_{RB}$, we can find a linear operator $V_{A\to B}$ such that $(I_R \otimes V_{A\to B}) |\Gamma\rangle_{RA} = |\phi\rangle_{RB}$. Consider also that

$$\begin{aligned}\langle i|_R |\phi\rangle_{RB} &= \langle i|_R (I_R \otimes V_{A\to B}) |\Gamma\rangle_{RA} &(18)\\ &= V_{A\to B} |i\rangle_A . &(19)\end{aligned}$$

Applying this to our case of interest, for each l, we can write

$$|\phi_l\rangle_{RB} = I_R \otimes (V_l)_{A\to B} |\Gamma\rangle_{RA} , \tag{20}$$

where $(V_l)_{A\to B}$ is some linear operator of the form in (13). After making this observation, we realize that it is possible to write

$$\begin{aligned}\mathcal{N}_{A\to B}(|i\rangle\langle j|) &= (\langle i|_R \otimes I_B)(\mathcal{N}_{A\to B}(|\Gamma\rangle\langle\Gamma|_{RA}))(|j\rangle_R \otimes I_B) &(21)\\ &= (\langle i|_R \otimes I_B) \sum_{l=0}^{d-1} |\phi_l\rangle\langle\phi_l|_{RB} (|j\rangle_R \otimes I_B) &(22)\\ &= \sum_{l=0}^{d-1} [(\langle i|_R \otimes I_B)|\phi_l\rangle_{RB}][\langle\phi_l|_{RB}(|j\rangle_R \otimes I_B)] &(23)\\ &= \sum_{l=0}^{d-1} V_l |i\rangle\langle j|_A V_l^\dagger . &(24)\end{aligned}$$

By linearity of the map $\mathcal{N}_{A\to B}$, it follows that its action on any operator σ can be written as follows:

$$\mathcal{N}_{A\to B}\left(\sigma\right)=\sum_{l=0}^{d-1}V_l\sigma V_l^\dagger, \tag{25}$$

since any operator σ can be written as a linear combination of operators in the basis $\{|i\rangle\langle j|\}$.

If the decomposition in (10) is a spectral decomposition, then it follows that the Kraus operators $\{V_l\}$ are orthogonal with respect to the Hilbert–Schmidt inner product:

$$\mathrm{Tr}\left\{V_l^\dagger V_k\right\}=\mathrm{Tr}\left\{V_l^\dagger V_l\right\}\delta_{l,k}. \tag{26}$$

This follows from the fact that

$$\begin{aligned}\delta_{l,k}\left\langle\phi_l|\phi_l\right\rangle &= \left\langle\phi_l|\phi_k\right\rangle &(27)\\ &=\left\langle\Gamma\right|_{RB}\left[I_R\otimes\left(V_l^\dagger V_k\right)_B\right]\left|\Gamma\right\rangle_{RB} &(28)\\ &=\mathrm{Tr}\left\{V_l^\dagger V_k\right\}. &(29)\end{aligned}$$

To prove the condition in (2), let us begin by exploiting the fact that the map $\mathcal{N}_{A\to B}$ is trace preserving, so that

$$\mathrm{Tr}\left\{\mathcal{N}_{A\to B}\left(|i\rangle\langle j|_A\right)\right\}=\mathrm{Tr}\left\{|i\rangle\langle j|_A\right\}=\delta_{ij}. \tag{30}$$

for all operators $\{|i\rangle\langle j|_A\}_{i,j}$. But consider also that

$$\begin{aligned}\mathrm{Tr}\left\{\mathcal{N}_{A\to B}\left(|i\rangle\langle j|_A\right)\right\} &= \mathrm{Tr}\left\{\sum_l V_l\left(|i\rangle\langle j|_A\right)V_l^\dagger\right\} &(31)\\ &=\mathrm{Tr}\left\{\sum_l V_l^\dagger V_l\left(|i\rangle\langle j|_A\right)\right\} &(32)\\ &=\langle j|_A\sum_l V_l^\dagger V_l\,|i\rangle_A\,. &(33)\end{aligned}$$

Thus, in order to have consistency with (30), we require that

$$\langle j|_A\sum_l V_l^\dagger V_l\,|i\rangle_A=\delta_{i,j}, \tag{34}$$

or equivalently, for (2) to hold. ■