Lecture 24

quantum error correction - the theory upon which practical quantum computation resides

There are a few problems to consider, that a priori seem like they might prohibit quantum error correction from working

1) no-cloning theorem - classical error correction codes introduce redundancy to mitigate the effects of errors. But we cannot copy quantum states...

2) measurement - in order to diagnose error, we need to look at bits, i.e., measure them in the classical case. But in QM this will disturb the state...

3) error can be continuous

Fortunately, there are ways around both problems.

1) use fresh ancilla qubits & copy basis (i.e., using CNOT for example)

2) perform measurements that learn only about errors & not the encoded information

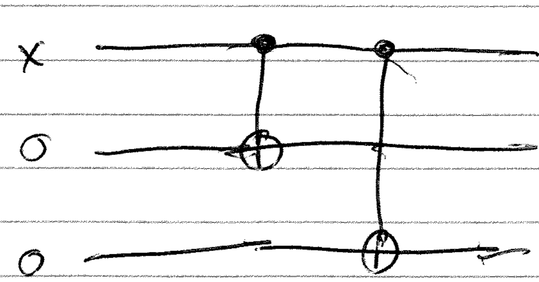
3) solving 3) is related to 2)

See this w/ an example: quantum repetition code

classical repetition code takes

0 → 000 $\qquad$ 1 → 111

could encode this using a reversible circuit as



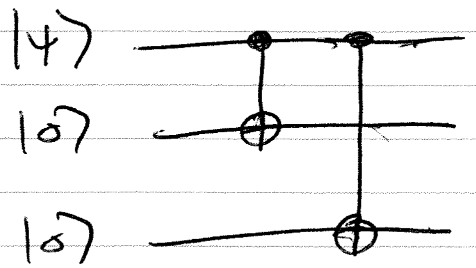
this code will protect against any single-bit error. i.e., decoding algorithm is

$$\left.\begin{matrix} 000 \\ 001 \\ 010 \\ 100 \end{matrix}\right\} \rightarrow 000 \rightarrow 0$$

$$\left.\begin{matrix} 110 \\ 101 \\ 011 \\ 111 \end{matrix}\right\} \rightarrow 111 \rightarrow 1$$

will decode incorrectly if there are 2 or 3-bit errors

First idea for generalizing to quantum case: use the same encoding circuit



state becomes

$$(\alpha|0\rangle + \beta|1\rangle)|0\rangle|0\rangle$$

$$\rightarrow \alpha|000\rangle + \beta|111\rangle$$

Now suppose a single Pauli X error happens on qubit 1, so that state becomes

$$\alpha|100\rangle + \beta|011\rangle$$

How to correct this error without learning anything about α or β?

Notice that by measuring the parity of neighboring qubits, we can learn about the error.

What is parity of 1st two qubits?
odd

& parity of last two?
even

Now suppose an error affects second qubit, taking it to

$\alpha|010\rangle + \beta|101\rangle$

parity measurements are odd, odd

error on third qubit gives

$\alpha|001\rangle + \beta|110\rangle$

& parity measurement gives even, odd

so here is our lookup table for error correction

Error	Syndrome
none	even, even
X_1	odd, even
X_2	odd, odd
X_3	even, odd

we have a unique syndrome for each error & can thus perform error correction in analogy to classical repetition code

How to actually perform the syndrome measurement?

If we could measure $Z \otimes Z$, this is a parity measurement because it can be written as

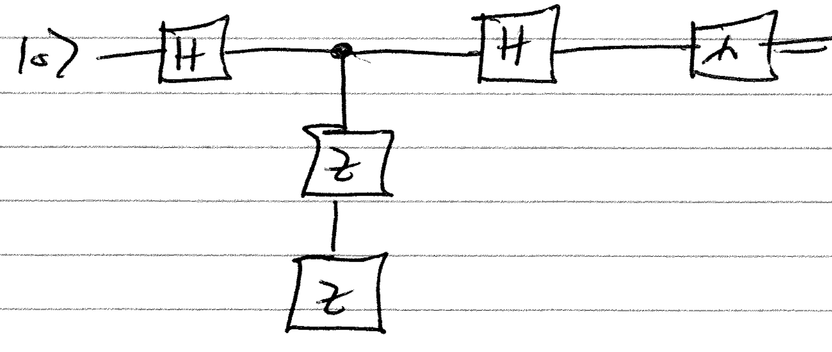
$$Z \otimes Z = |00\rangle\langle 00| + |11\rangle\langle 11| - \left(|01\rangle\langle 01| + |10\rangle\langle 10| \right)$$

eigenvalue $+1 \Leftrightarrow$ even parity

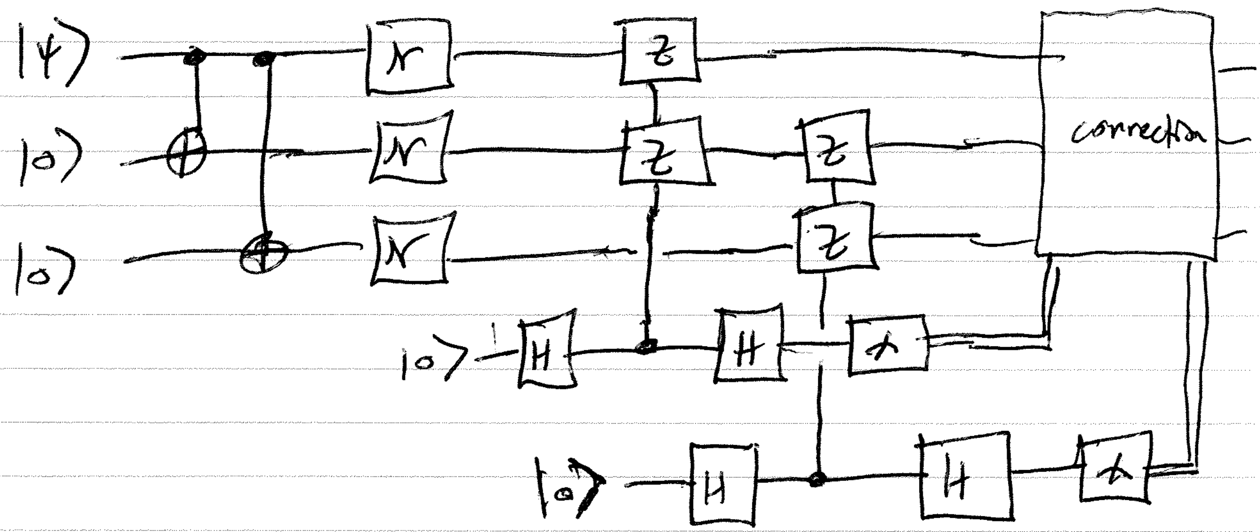
" $-1 \Leftrightarrow$ odd parity

How to measure these eigenvalues?
I.e., what is a q. circuit to do so?

Use phase estimation algorithm.



So the full circuit for error correction would look like



After correction, you might wish to decode, but in computation, it can be helpful to keep the quantum information encoded. How to compute on the encoded data? It can be helpful to determine the logical operators of the code.

What are the logical operators?

Consider that

$$\text{CNOT}\,(X \otimes I)\,\text{CNOT}^\dagger = X \otimes X$$

$$\text{''}\quad (Z \otimes I)\ \cancel{\text{CNOT}^\dagger}\ \text{''} = Z \otimes I$$

$$\text{''}\quad (I \otimes X)\quad \text{''} = I \otimes X$$

$$\text{''}\quad (I \otimes Z)\quad \text{''} = Z \otimes Z$$

CNOT propagates X forward & Z backward

Then encoded state is

$$\text{CNOT}_{1\to 2}\,\text{CNOT}_{1\to 3}\,|\psi\rangle|0\rangle|0\rangle = |\bar{\psi}\rangle$$

What is the operation that realizes a logical X operation?

Consider that

$$\text{CNOT}_{1\to 2}\,\text{CNOT}_{1\to 3}\,X_1\,|\psi\rangle|0\rangle|0\rangle$$

$$= \underbrace{\text{CNOT}_{1\to 2}\,\text{CNOT}_{1\to 3}\,X_1\,\text{CNOT}^\dagger_{1\to 3}\,\text{CNOT}^\dagger_{1\to 2}}_{X \otimes X \otimes X}\,\text{CNOT}_{1\to 2}\,\text{CNOT}_{1\to 3}\,|\psi\rangle|0\rangle|0\rangle$$

$$X \otimes X \otimes X\ |\bar{\psi}\rangle$$

What about logical Z?

$|\bar{\Psi}\rangle$ Set $U = CNOT_{1\to 2}\, CNOT_{1\to 3}$

Consider that

$$U\, Z_1 |\psi\rangle|00\rangle = U Z U^{\dagger} U |\psi\rangle|00\rangle$$

$$\equiv \bar{Z}\, |\bar{\Psi}\rangle$$

so $\bar{Z} = Z \otimes I \otimes I$

Also, observe that

$$Z_2 |\psi\rangle|0\rangle|0\rangle = |\psi\rangle|0\rangle|0\rangle$$

$$Z_2 Z_3 |\psi\rangle|0\rangle|0\rangle = \text{"}$$

so Z_2 + $Z_2 Z_3$ stabilize unencoded quantum state

then

$$U Z_2 U^{\dagger} U |\psi\rangle|0\rangle|0\rangle = Z_1 Z_2 |\bar{\Psi}\rangle$$

measurement operator

$$U\, Z_2 Z_3 U^{\dagger} U\, |\psi\rangle|0\rangle|0\rangle$$

$$= U Z_2 U^{\dagger}\, U Z_3 U^{\dagger} U\, |\psi\rangle|0\rangle|0\rangle$$

$$= (Z_1 Z_2)(Z_1 Z_3)\, |\bar{\Psi}\rangle = Z_2 Z_3 |\bar{\Psi}\rangle$$

another measurement operator

What if a continuous error occurs?

such as $e^{iX_1\theta}$?

consider that

$$e^{iX_1\theta} = \cos\theta\, I + i\sin\theta\, X_1$$

$$\Rightarrow e^{iX_1\theta}|\Psi\rangle = \cos\theta\,|\Psi\rangle + i\sin\theta\, X_1|\Psi\rangle$$

Now if we then measure $Z_1Z_2 + Z_2Z_3$,

state collapses to

$|\Psi\rangle$ w/ probability $\cos^2\theta$ & outcomes are $+1, +1$

or to

$X_1|\Psi\rangle$ w/ probability $\sin^2\theta$ & outcomes are $-1, +1$

So this solves the third problem & we see that the process of measurement digitizes the error

The quantum repetition code only corrects particular errors. How to correct more general ones?

Consider that any quantum channel $\mathcal{N}$ can be "twirled" to a Pauli channel, i.e.,

$$\underbrace{\frac{1}{|\mathcal{P}_n|}\sum_{V \in \mathcal{P}_n} V^\dagger \mathcal{N}(V \rho V^\dagger) V}_{\text{this channel is a Pauli channel.}}$$

So then it suffices to build codes that correct Pauli errors.

Encoder is U chosen from Clifford group (those unitaries that preserve the Pauli group) then encoded state is

$$\mathcal{U}\,|\psi\rangle_k\,|0\rangle^{\otimes n-k} = |\bar{\psi}\rangle$$

encodes k qubits into n qubits.

Operators to measure to correct errors are given by $\bar{Z}_i$ for $i \in \{1, \dots, n-k\}$ where these are defined by

$$|\bar{\psi}\rangle = \mathcal{U}\, Z_i\, |\psi\rangle_k\, |0\rangle^{\otimes n-k}$$

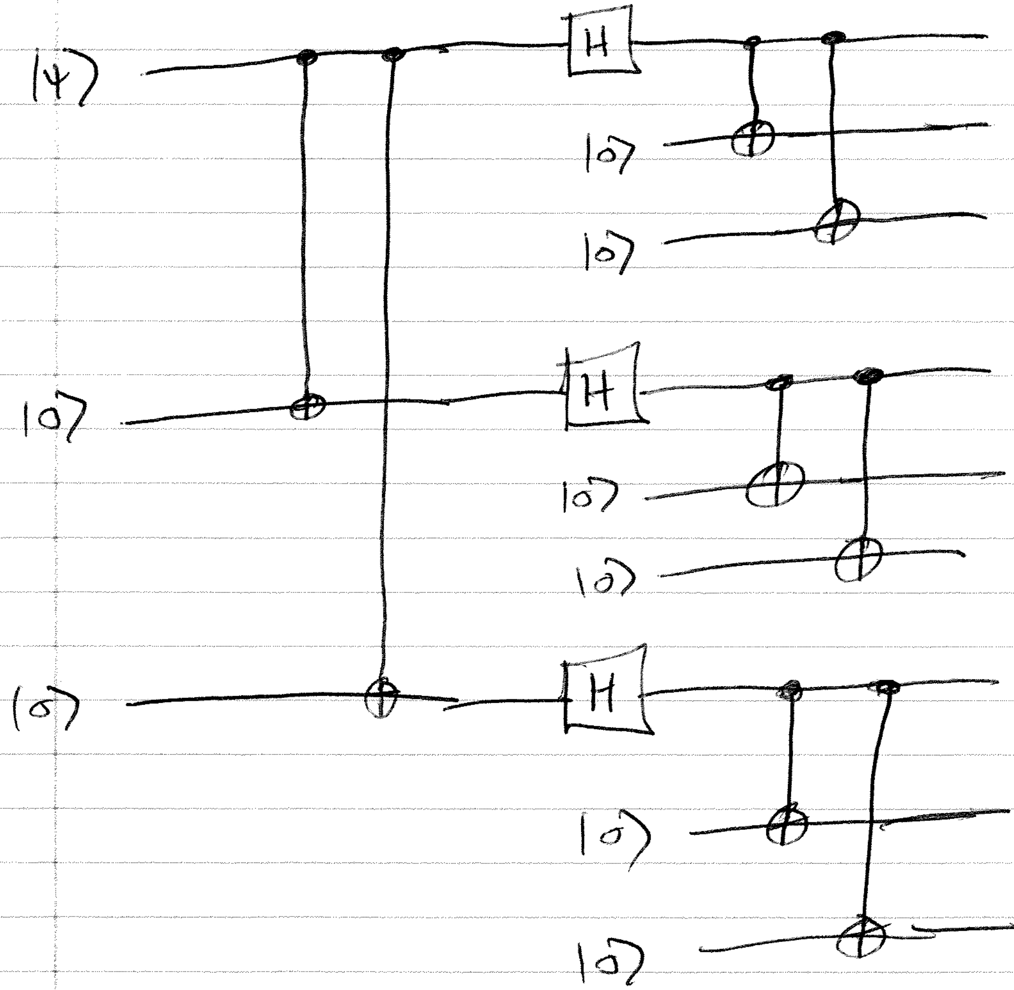
$$= \mathcal{U} Z_i \mathcal{U}^{+}\; \mathcal{U}\, |\psi\rangle_k\, |0\rangle^{\otimes n-k}$$

$$\equiv \bar{Z}_i\, |\bar{\psi}\rangle$$

a code that corrects t ~~[illegible]~~ -qubit errors is said to have distance $2t+1 = d$ & be an $[n, k, d]$ code.

* mention about cluster/stabilizer states & Gottesman-Knill theorem *

Example of a code that corrects a single-qubit errors is known as Shor code. It can be understood as a concatenation of a repetition code w/a phase repetition code. Encoding circuit is



Can work out that a set of ~~the~~ stabilizer operators are

$Z_1 Z_2$, $Z_1 Z_3$, $Z_4 Z_5$, $Z_4 Z_6$, $Z_7 Z_8$, $Z_7 Z_9$,

$X_1 \cdots X_6$, $X_1 X_2 X_3 X_7 X_8 X_9$

& these correct a single-qubit error by assigning a unique syndrome to each of the errors.

Smallest code that corrects a single-qubit error is

$X_1 Z_2 Z_3 X_4$,

$X_2 Z_3 Z_4 X_5$,

$X_1 X_3 Z_4 Z_5$,

$Z_1 X_2 X_4 Z_5$ w/ logical operators

$\overline{X} = X_1 \cdots X_5$

$\overline{Z} = Z_1 \cdots Z_5$

- can build a theory of fault tolerance for Q.C. from the theory of q. error correction

- need to show how every q. gate can be replaced by a fault tolerant implementation.

- main theorem is that as long as error per gate is small enough, then arbitrarily long q. computations are possible.