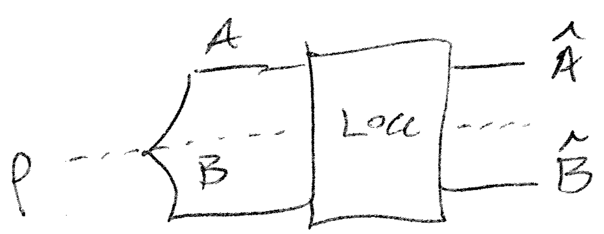
$\boxed{\text{Lecture 29}}$

<u>Entanglement Distillation</u>

one-shot setting:



ED protocol defined by

$\mathcal{L} \in \text{LOCC}$ & d &

error prob. is

$$p_{err} = 1 - F(\Phi^d_{\hat{A}\hat{B}}, \mathcal{L}_{AB\to\hat{A}\hat{B}}(\rho_{AB}))$$

$$= \text{Tr}\{(I_{\hat{A}\hat{B}} - \Phi^d_{\hat{A}\hat{B}})\mathcal{L}_{AB\to\hat{A}\hat{B}}(\rho_{AB})\}$$

"probability that final state would fail a test that ~~it~~ is $\Phi^d_{\hat{A}\hat{B}}$"

2

one-shot distillable entanglement of ρ_{AB}:

$$E_D^{\varepsilon}(A;B)_\rho = \sup_{(d,\,\mathcal{R} \in LOCC)} \{\log_2 d : p_{err} \leq \varepsilon\}$$

Upper bound:

can place an upper bound on this quantity by using Rains relative entropy

Key question:

Which states are useless for entanglement distillation?

separable states!

why? because they pass the entanglement test w/ a very low probability related to dimension:

$$\mathrm{Tr}[\Phi^d_{\hat{A}\hat{B}}\, \sigma_{\hat{A}\hat{B}}] \leq \frac{1}{d}$$

More generally, we can consider the set

$$PPT'(A:B) = \{\sigma_{AB} : \sigma_{AB} \leq 0, \; \| T_B(\sigma_{AB}) \|_1 \leq 1\}$$

which contains SEP.

Then consider that

$$\mathrm{Tr}[\Phi^d_{AB}\, \sigma_{AB}] \leq \frac{1}{d} \quad \forall\, \sigma_{AB} \in PPT'$$

<u>Proof:</u>

$$\mathrm{Tr}\{\Phi^d_{AB}\, \sigma_{AB}\} = \mathrm{Tr}\{T_B(\Phi^d_{AB})\, T_B(\sigma_{AB})\}$$

unitary swap operator

$$= \frac{1}{d}\,\mathrm{Tr}\{F_{AB}\, T_B(\sigma_{AB})\}$$

$$\leq \frac{1}{d}\, \| T_B(\sigma_{AB}) \|_1$$

$$\leq \frac{1}{d}$$

Define the ε - Rains relative entropy as

$$R_H^\varepsilon(A;B)_\rho = \inf_{\sigma_{AB} \in PPT'(A;B)} D_H^\varepsilon(\rho_{AB} \| \sigma_{AB})$$

Then

$$E_D^\varepsilon(A;B)_\rho \leq R_H^\varepsilon(A;B)_\rho$$

Proof: Consider an arbitrary ED protocol w/ initial state ρ_{AB} + final state

$$\omega_{\hat{A}\hat{B}} = \mathcal{L}_{AB \to \hat{A}\hat{B}}(\rho_{AB})$$

Then $R_H^\varepsilon(\hat{A};\hat{B})_\omega \leq R_H^\varepsilon(A;B)_\rho$

b/c

$$D_H^\varepsilon(\omega_{\hat{A}\hat{B}} \| \underbrace{\mathcal{L}_{AB \to \hat{A}\hat{B}}(\sigma_{AB})}_{\in PPT'}) \leq D_H^\varepsilon(\rho_{AB} \| \sigma_{AB})$$

D.P. +

The final state $\omega_{\hat{A}\hat{B}}$ satisfies

$$\mathrm{Tr}\left[\Phi^d_{\hat{A}\hat{B}}\, \omega_{\hat{A}\hat{B}}\right] \geq 1 - \varepsilon$$

& we also have that

$$\mathrm{Tr}\left[\Phi^d_{\hat{A}\hat{B}}\, \sigma_{\hat{A}\hat{B}}\right] \leq \frac{1}{d} \quad \forall\, \sigma_{\hat{A}\hat{B}} \in PPT'$$

By definition of HTRE, it then follows that

$$\log_2 d \leq D_H^{\varepsilon}(\omega_{\hat{A}\hat{B}} \| \sigma_{\hat{A}\hat{B}}) \quad \forall\, \sigma_{\hat{A}\hat{B}} \in PPT'$$

$$\Rightarrow \log_2 d \leq R_H^{\varepsilon}(\hat{A};\hat{B})_{\omega}$$
$$\leq R_H^{\varepsilon}(A;B)_{\rho}$$

Since this holds $\forall$ protocols, we conclude that

$$E_D^{\varepsilon}(A;B)_{\rho} \leq R_H^{\varepsilon}(A;B)_{\rho}$$

~~Bonus~~ Bonus that R_H^ε can be computed by a semi-definite program.

Can then get the bound

$$E_D^\varepsilon(A;B) \leq \widetilde{R}_\alpha(A;B)_\rho + \frac{\alpha}{\alpha-1}\log\left(\frac{1}{1-\varepsilon}\right)$$

$\forall \alpha > 1$

using usual technique from previous lectures.

can also get

$$E_D^\varepsilon(A;B)_\rho \leq \frac{1}{1-\varepsilon}\left[R(A;B)_\rho + h_2(\varepsilon)\right]$$

---

using similar ideas, we can get the bound

$$E_D^\varepsilon(A;B)_\rho \leq \frac{1}{1-\sqrt{\varepsilon}}\left(E_{sq}(A;B)_\rho + g_2(\sqrt{\varepsilon})\right)$$

these are the two best known upper bounds on distillable entanglement.

Now let us give a lower bound on distillable entanglement.

Asymptotic lower bound is

$$E_D(\rho_{AB}) \geq I(A\rangle B)_\rho = H(B)_\rho - H(AB)_\rho$$

coherent information

Define some quantities

$$D_{max}(\rho\|\sigma) = \log_2 \|\sigma^{-1/2}\rho\,\sigma^{-1/2}\|_\infty$$

$$= \inf_\lambda \{\lambda : \rho \leq 2^\lambda \sigma\}$$

conditional min-entropy

$$H_{min}(A|B)_\rho = -\inf_{\sigma_B} D_{max}(\rho_{AB}\|I_A \otimes \sigma_B)$$

Define $B^{\varepsilon}(\rho) = \{\tilde{\rho} : \sqrt{1-F(\rho,\tilde{\rho})} \leq \varepsilon\}$

Smooth cond. min-entropy

$$H_{min}^{\varepsilon}(A|B)_{\rho} = \sup_{\tilde{\rho} \in B^{\varepsilon}(\rho)} H_{min}(A|B)_{\tilde{\rho}}$$

<u>Thm</u>: For a state ρ_{AB}, $\varepsilon \in (0,1]$,

$\eta \in [0, \sqrt{\varepsilon})$,

$$E_D^{\varepsilon}(A;B)_{\rho} \geq H_{min}^{\sqrt{\varepsilon}-\eta}(A|E)_{\psi} + 4\log_2 \eta$$

where ψ_{ABE} purifies ρ_{AB}.

<u>Basic idea of proof</u>:

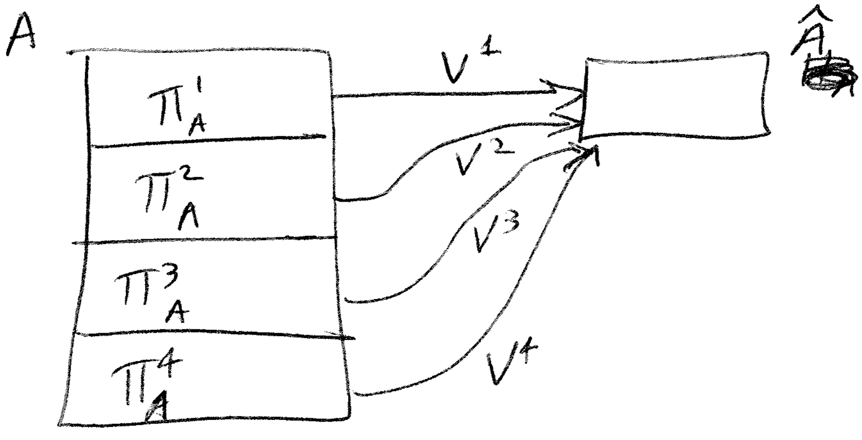
Apply a random unitary U_A to ψ_{ABE}, giving $U_A \psi_{ABE} U_A^{\dagger}$

Then apply the channel

$$\mathcal{E}_{A \to \hat{A} X_A}(\cdot) = \sum_x V^x_{A\to\hat{A}} \Pi^x_A (\cdot) \Pi^x_A V^{x\dagger}_{A\to\hat{A}} \otimes |x\rangle\langle x|_{X_A}.$$

Π^x_A is a projector & $\sum_x \Pi^x_A = I_A$

each Π^x_A projects onto space of size $d_{\hat{A}}$. Then $V^x_{A\to\hat{A}}$ is an isometry that embeds into $\mathcal{H}_{\hat{A}}$



can prove that $\exists\, U_A$ such that

$$\sum_x \sqrt{\frac{p(x)}{d_{X_A}}} \sqrt{F}\left(\omega^x_{\hat{A}E}, \pi_{\hat{A}} \otimes \tilde{\rho}_E\right) \geq \sqrt{1-\varepsilon}$$

where $p(x) = \mathrm{Tr}[\Pi^x_A \rho_A]$ $\quad U_A \quad U_A^\dagger$

$$\omega^x_{\hat{A}E} = \frac{1}{p(x)} V^x_{A\to\hat{A}} \Pi^x_A \psi_{ABE} \Pi^x_A (V^x_{A\to\hat{A}})^\dagger$$

$U_A \quad \mathrm{Tr}_B[\cdot] \quad U_A^\dagger$

This unitary U_A is called a decoupling unitary b/c it decouples A from E.

<u>Recall Uhlmann's thm:</u>

$$\psi^x_{\hat{A}BE} = \frac{1}{p(x)} V^x_{A\to\hat{A}} \Pi^x_A U_A \psi_{ABE} U_A^\dagger \Pi^x_A V^{x\dagger}$$

purifies $\omega^x_{\hat{A}E}$

$\Phi_{\hat{A}\hat{B}} \otimes \tilde{\varphi}_{B'E}$ purifies $\pi_{\hat{A}} \otimes \tilde{\rho}_E$

$\exists$ isometric channel $\mathcal{W}^x_{B\to\hat{B}B'}$ such that

$$F(\omega^x_{\hat{A}E}, \pi_{\hat{A}} \otimes \tilde{\rho}_E)$$

$$= F(\mathcal{W}^x_{B\to\hat{B}B'}(\psi^x_{\hat{A}BE}), \Phi_{\hat{A}\hat{B}} \otimes \tilde{\varphi}_{B'E})$$

$\mathcal{D}^x_{B\to\hat{B}} =$ $\mathrm{Tr}_{B'} \circ \mathcal{W}^x$ is Bob's decoding channel!

Then overall ~~2~~ 1 way-LOCC channel is

$$\sum_x \mathcal{E}^x_{A\to\hat{A}} \otimes \mathcal{D}^x_{B\to\hat{B}}$$

+ achieves desired error