Lecture 26

In this lecture, we establish a lower bound on one-shot entanglement-assisted capacity:

$$C_{EA}^{\varepsilon}(\mathcal{N}) \geq \overline{I}_H^{\varepsilon-\eta}(\mathcal{N}) - \log_2\left(\frac{4\varepsilon}{\eta^2}\right)$$

where $\eta \in (0, \varepsilon)$

& $\overline{I}_H^{\varepsilon}(\mathcal{N}) = \sup_{\psi_{RA}} \overline{I}_H^{\varepsilon}(R;B)_\omega$

$$\omega_{RB} = \mathcal{N}_{A\to B}(\psi_{RA})$$

&

$$\overline{I}_H^{\varepsilon}(A;B)_\rho = D_H^{\varepsilon}(\rho_{AB} \| \rho_A \otimes \rho_B)$$

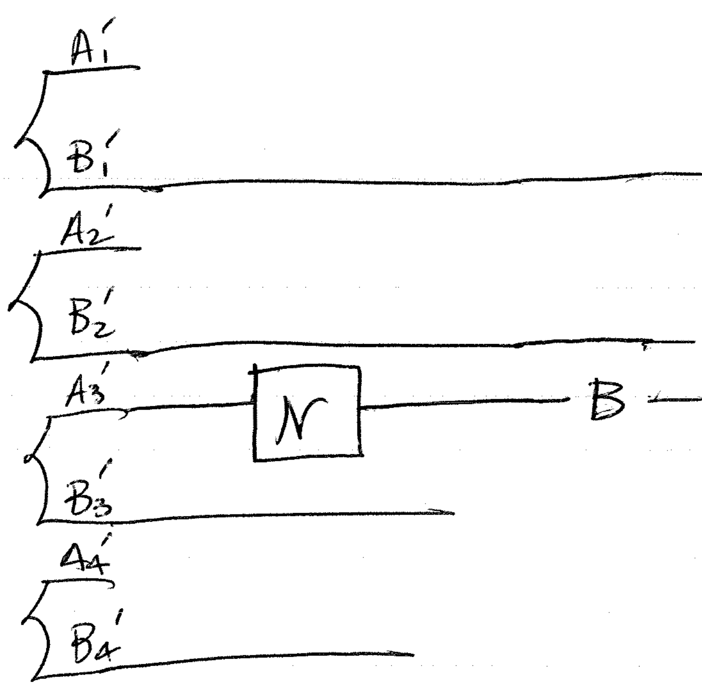
The method used is called <u>position-based coding</u> & <u>sequential decoding</u>

The basic idea behind the comm. scheme is simple to illustrate:

Let $\rho_{A'B'}$ be a resource state, & let A & B share $|M|$ copies of it, where $|M|$ is the # of messages:

$$\rho_{A'B'}^{\otimes |M|}$$

Suppose, e.g., that $|M| = 4$.
Suppose Alice wants to send message 3.
Then scheme looks like



Consider the reduced state of systems B_1' & B:

$$\rho_{B_1'} \otimes \mathcal{N}_{A\to B}(\rho_{A_3'}) = \rho_{B'} \otimes \mathcal{N}_{A\to B}(\rho_{A'})$$

reduced state of B_2' & B:

$$\rho_{B_2'} \otimes \mathcal{N}_{A\to B}(\rho_{A_3'}) = \rho_{B'} \otimes \mathcal{N}_{A\to B}(\rho_{A'})$$

reduced state of B_4' & B:

$$\rho_{B_4'} \otimes \mathcal{N}_{A\to B}(\rho_{A_3'}) = \rho_{B'} \otimes \mathcal{N}_{A\to B}(\rho_{A'})$$

reduced state of B_3' & B:

$$\mathcal{N}_{A\to B}(\rho_{B_3' A_3'}) = \mathcal{N}_{A\to B}(\rho_{B'A'}) \qquad \text{Bingo!}$$

<u>Idea</u>: suppose we have a measurement operator $\Lambda_{B'B}$ that can distinguish the state $\mathcal{N}_{A\to B}(\rho_{B'A'})$ from the product state $\rho_{B'} \otimes \mathcal{N}_{A\to B}(\rho_{A'})$

then, conceptually, Bob could perform the measurement $\{\Lambda, I-\Lambda\}$ sequentially on $B_1'B$, then $B_2'B$, then $B_3'B$, etc. until he gets the outcome corresponding to a correlated state, rather than a product state.

- Main obstacle to overcome is that measurement disturbs a state (in particular, the system B, which is repeatedly measured)

Let us now discuss a proof.

Let $\Lambda_{B'B}$ be an optimal measurement for the hypo. testing relative entropy:

$$D_H^{\varepsilon-\eta}\left(N_{A\to B}(\rho_{B'A'}) \,\|\, \rho_{B'} \otimes N_{A\to B}(\rho_{A'})\right)$$

$$= -\log \inf_{\Lambda_{B'B}\geq 0} \Big\{ \mathrm{Tr}\left[\Lambda_{B'B}\left(\rho_{B'} \otimes N_{A\to B}(\rho_{A'})\right)\right] : \Lambda_{B'B} \leq I_{B'B},\ \mathrm{Tr}\left[\Lambda_{B'B} N_{A\to B}(\rho_{B'A'})\right] \geq 1-(\varepsilon-\eta) \Big\}$$

this measurement can be implemented using Naimark extension theorem as

$$\{ \Pi_{B'BR},\ I_{B'BR} - \Pi_{B'BR} \}$$

where

$$\Pi_{B'BR} = U^{\dagger}_{B'BR} \left(I_{B'B} \otimes |1\rangle\langle 1|_R\right) U_{B'BR}$$

$$U_{B'BR} = \sqrt{I_{B'B} - \Lambda_{B'B}} \otimes I_R + \sqrt{\Lambda_{B'B}} \otimes \sigma_X \sigma_Z$$

Then $\mathrm{Tr}\left[\Pi_{B'BR}\left(\mathcal{N}_{A\to B}(\rho_{B'A'})\otimes|0\rangle\langle 0|_R\right)\right]$

$$= \mathrm{Tr}\left[\Lambda_{B'B}\,\mathcal{N}_{A\to B}(\rho_{B'A'})\right]$$

$$\geq 1-\varepsilon \quad \& \quad -\log_2 \mathrm{Tr}\left[\Lambda_{B'B}\left(\rho_{B'}\otimes\mathcal{N}(\rho_{A'})\right)\right] = I_H^{\varepsilon-\eta}(B';B)_{\omega}$$

w/ $\omega_{B'B} = \mathcal{N}_{A\to B}(\rho_{A'B'})$

Now define $P_i = \Pi_{B_i'BR_i}$

$\&\ \hat{P}_i = I - P_i$

Bob performs sequential decoding

for $i = 1, \ldots, |\mathcal{M}|$

measure $\{P_i, \hat{P}_i\}$

until the first outcome is obtained

probability of guessing msg. m' when m is sent is then

$$q(m'|m) = \mathrm{Tr}\left[P_{m'}\hat{P}_{m'-1}\cdots\hat{P}_1\,\omega^m_{B_1'\cdots B_{|\mathcal{M}|}'R_1\cdots R_{|\mathcal{M}|}}\times \hat{P}_1\cdots\hat{P}_{m'-1}P_{m'}\right]$$

where

$$\omega^m_{\cdots} = \rho_{B_1'} \otimes \cdots \otimes \mathcal{N}_{A\to B}(\rho_{AB_m'}) \otimes \cdots \otimes \rho_{B'_{|\mathcal{M}|}}$$
$$\otimes |0,\ldots,0\rangle\langle 0,\ldots,0|_{R_1\cdots R_{|\mathcal{M}|}}$$

$\Rightarrow$ error prob. for msg. m is

$$1 - \mathrm{Tr}[P_m \hat{P}_{m-1} \cdots \hat{P}_1 \omega^m \hat{P}_1 \cdots \hat{P}_{m-1} P_m]$$

Lemma: there is a q. generalization of the union bound.

~~Let $\{P_i\}$~~

Applying it, we get the upper bound

$$\leq (1+c)\,\mathrm{Tr}[\hat{P}_m \omega^m] + (2+c+c^{-1}) \sum_{i=1}^{m-1} \mathrm{Tr}[P_i \omega^m]$$

holds $\forall c>0$.

$$= (1+c)\,\mathrm{Tr}\{(I - \Lambda_{B'B})(\mathcal{N}_{A\to B}(\rho_{B'A'}))\} + (2+c+c^{-1})(m-1)\,\mathrm{Tr}[\Lambda_{B'B}(\rho_{B'} \otimes \mathcal{N}_{A\to B}(\rho_{A'}))]$$

$$\leq (1+c)(\varepsilon-\eta) + (2+c+c^{-1}) \times$$
$$|\mathcal{M}|\, 2^{-\bar{I}_H^{\varepsilon-\eta}(B';B)_\rho}$$

Now pick $c = \dfrac{\eta}{2\varepsilon-\eta}$ &

$$|\mathcal{M}| = 2^{\bar{I}_H^{\varepsilon-\eta}(B';B)_\rho - \log_2\left(\frac{4\varepsilon}{\eta^2}\right)}$$

$\Rightarrow$ last line $=$

$$\varepsilon\left(\frac{2\varepsilon-(2-\varepsilon^2)\eta}{2\varepsilon-\eta}\right) \leq \varepsilon$$

$$\Rightarrow \quad p_{err}(m) \leq \varepsilon \quad \forall m \in \mathcal{M}$$

$$\Rightarrow \quad p^*_{err} \leq \varepsilon$$

this holds for all entangled states. Now optimize over all of them & set

$$\log_2 |\mathcal{M}| = \sup_{\rho_{A'B'}} \bar{I}_H^{\varepsilon-\eta}(B';B) - \log_2\left(\frac{4\varepsilon}{\eta^2}\right)$$

$$= \bar{I}_H^{\varepsilon-\eta}(\mathcal{N}) - \text{"}$$

suffices to take $\rho_{A'B'}$ to

be a pure state.

Putting everything together, we have proved that

$$C_{EA}^{\varepsilon}(\mathcal{N}) \geq \bar{I}_H^{\varepsilon-\eta}(\mathcal{N}) - \log_2\left(\frac{4\varepsilon}{\eta^2}\right)$$

lower bound on one-shot

EA capacity

Now using the lower bound

$$D_H^{\varepsilon}(\rho\|\sigma) \geq D_\alpha(\rho\|\sigma) + \frac{\alpha}{\alpha-1}\log_2\left(\frac{1}{\varepsilon}\right)$$

$\forall \alpha, \varepsilon \in (0,1)$

$\Rightarrow$

$$C_{EA}^{\varepsilon}(\mathcal{N}) \geq \bar{I}_\alpha(\mathcal{N}) + \frac{\alpha}{\alpha-1}\log_2\left(\frac{1}{\varepsilon-\eta}\right) - \log\left(\frac{4\varepsilon}{\eta^2}\right)$$

$\forall \varepsilon, \alpha \in (0,1), \eta \in (0,\varepsilon)$

where $\bar{I}_\alpha(\mathcal{N}) = \sup_{\psi_{RA}} \bar{I}_\alpha(R;B)_\omega$

$$\omega_{RB} = \mathcal{N}_{A\to B}(\psi_{RA})$$

$$\bar{I}_\alpha(A;B)_\rho = D_\alpha(\rho_{AB}\|\rho_A \otimes \rho_B)$$

Asymptotic EA capacity:

$$C_{EA}(\mathcal{N}) = \inf_{\varepsilon \in (0,1)} \liminf_{n \to \infty} \frac{1}{n} C_{EA}^{\varepsilon}(\mathcal{N}^{\otimes n})$$

we then get

$$\frac{1}{n} C_{EA}^{\varepsilon}(\mathcal{N}^{\otimes n}) \geq \frac{1}{n} \overline{I}_{\alpha}(\mathcal{N}^{\otimes n}) + \frac{\alpha}{n(\alpha - 1)} \log\left(\frac{1}{\varepsilon - \eta}\right) - \frac{1}{n} \log_2\left(\frac{4\varepsilon}{\eta^2}\right)$$

$$\geq \overline{I}_{\alpha}(\mathcal{N}) + \cdots$$

$$\Rightarrow \liminf_{n \to \infty} \frac{1}{n} C_{EA}^{\varepsilon}(\mathcal{N}^{\otimes n}) \geq \overline{I}_{\alpha}(\mathcal{N}) \quad \forall \alpha \in (0,1)$$

take limit as $\alpha \to 1$

$$\Rightarrow \liminf_{n \to \infty} \frac{1}{n} C_{EA}^{\varepsilon}(\mathcal{N}^{\otimes n}) \geq I(\mathcal{N})$$

where $I(\mathcal{N}) = \sup_{\psi_{RA}} I(R;B)_{\omega}$

then taking inf over $\epsilon \in (0,1)$

gives

$C_{EA}(N) \geq I(N)$

combined w/ converse from last time, we conclude

$$\boxed{C_{EA}(N) = I(N)}$$