Lecture 22

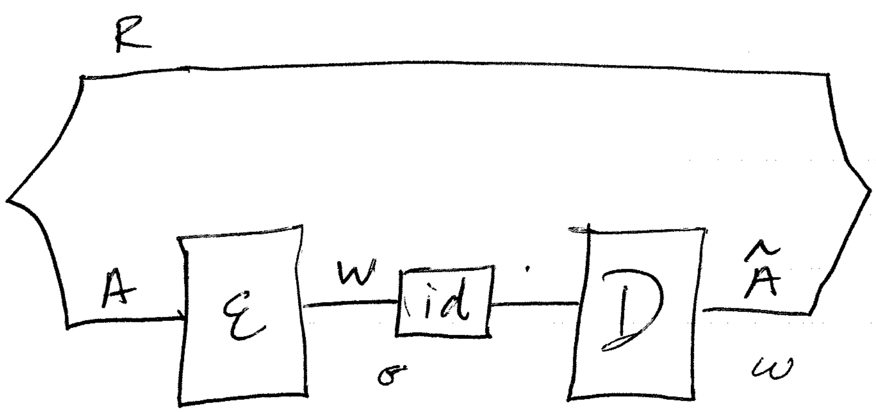
quantum data compression

(AKA Schumacher compression)

one-shot version of a q. data compression protocol.

Given is a density operator ρ_A w/ purification φ^{ρ}_{RA}.

General protocol is as follows:



where $\mathcal{E}$ is an encoding or compression channel & $\mathcal{D}$ is a decoding or decompression channel

Protocol is an $(|W|, \varepsilon)$ protocol

if $F(\varphi^{\rho}_{R\hat{A}}, (\mathcal{D}_{W\to\hat{A}} \circ \mathcal{E}_{A\to W})(\varphi^{\rho}_{RA})) \geq 1-\varepsilon$ (*)

one-shot ε-data compression limit

$$QDL^{\varepsilon}(\rho) = \inf_{(\mathcal{E},\mathcal{D}),|W|} \{\log_2 |W| : (*) \text{ holds}\}$$

<u>Asymptotic quantity</u>

$$QDL(\rho) = \sup_{\varepsilon\in(0,1)} \limsup_{n\to\infty} \frac{QDL^{\varepsilon}(\rho^{\otimes n})}{n}$$

<u>Strong converse quantity</u>

$$\widetilde{QDL}(\rho) = \inf_{\varepsilon\in(0,1)} \liminf_{n\to\infty} \frac{QDL^{\varepsilon}(\rho^{\otimes n})}{n}$$

We always have

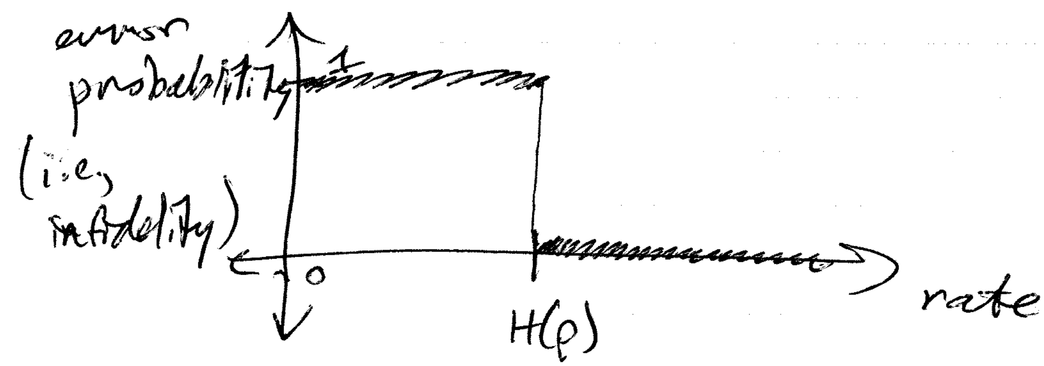
$$QDL(\rho) \geq \widetilde{QDL}(\rho)$$

Schumacher Data compression theorem

$$QDL(\rho) = \widetilde{QDL}(\rho) = H(\rho)$$

(strong converse due to Winter)

visualization of this theorem in $n \to \infty$ limit



above data compression limit, there exists a sequence of protocols w/ error prob. tending to zero.

below QDL, every scheme has error probability tending to one.

So QDL is a sharp dividing line between possible & impossible.

Let us start w/ strong converse.

Key tool is Renyi mutual information

$$\tilde{I}_\alpha(C;D)_\rho = \inf_{\sigma_D} \tilde{D}_\alpha(\rho_{CD} \| \rho_C \otimes \sigma_D)$$

~~Propert~~ where

$$\tilde{D}_\alpha(\omega \| \tau) = \frac{1}{\alpha-1} \log_2 \mathrm{Tr}\left[\left(\tau^{\frac{1-\alpha}{2\alpha}} \omega \tau^{\frac{1-\alpha}{2\alpha}}\right)^\alpha\right]$$

<u>Key properties</u>

$$\tilde{I}_\alpha(C;D)_\rho \leq 2\log_2|D| \quad \text{for } \alpha \in [1/2, 1) \cup (1,\infty)$$

Data processing under local channels

$$\tilde{I}_\alpha(C;D)_\rho \geq \tilde{I}_\alpha(C';D')_\omega$$

where

$$\omega_{C'D'} = (\mathcal{N}_{C\to C'} \otimes \mathcal{M}_{D\to D'})(\rho_{CD})$$

pseudo-continuity:

$$\tilde{I}_\beta(C;D)_\rho - \tilde{I}_\alpha(C;D)_\sigma \geq \frac{\alpha}{1-\alpha} \log F(\rho_{CD}, \sigma_{CD})$$

for $\alpha \in (1/2, 1)$ & $\beta = \frac{\alpha}{2\alpha-1}$

Proofs: see arXiv:1506.02635

Also:

$$\tilde{I}_\alpha(C;D)_\psi = 2 H_{\frac{1}{2\alpha-1}}(C)_\psi$$

for pure state ψ_{CD} &

$\alpha \in (0,1) \cup (1, \infty)$

in PQCT book

$\alpha \in (1/2, 1) \quad \beta = \frac{\alpha}{2\alpha - 1}$

Now use all these properties to prove a one-shot bound:

for an arbitrary $(|W|, \varepsilon)$ protocol

$$2 \log_2 |W| \geq \tilde{I}_\beta (R; W)_\sigma \quad \text{(dim. bound)}$$

$$\geq \tilde{I}_\beta (R; \hat{A})_\omega \quad \text{(data processing)}$$

$$\geq \tilde{I}_\alpha (R; \hat{A})_{\varphi^\rho} \quad \text{(pseudo-continuity)}$$

$$+ \frac{\alpha}{1-\alpha} \log_2 F(\varphi^\rho_{R\hat{A}}, \omega_{R\hat{A}})$$

$$\geq \tilde{I}_\alpha (R; \hat{A})_{\varphi^\rho}$$

$$+ \frac{\alpha}{1-\alpha} \log_2 (1 - \varepsilon)$$

$$= 2 H_{\frac{1}{2\alpha-1}} (A)_\rho + \frac{\alpha}{1-\alpha} \log_2 (1-\varepsilon)$$

$$\Rightarrow \log_2 |W| \geq H_{\frac{1}{2\alpha-1}} (A)_\rho + \frac{\alpha}{2(1-\alpha)} \log_2 (1-\varepsilon)$$

holds for arbitrary protocol

$\Rightarrow$ one-shot bound

$$QDL^\varepsilon(\rho) \geq H_{\frac{1}{2\alpha-1}} (A)_\rho + \frac{\alpha}{2(1-\alpha)} \log_2 (1-\varepsilon)$$

IID case:

$$\frac{1}{n} QDL^{\varepsilon}(\rho^{\otimes n})$$

$$\geq \frac{1}{n} H_{\frac{1}{2\alpha-1}}(\rho^{\otimes n}) + \frac{\alpha}{2n(1-\alpha)} \log_2(1-\varepsilon)$$

$$= H_{\frac{1}{2\alpha-1}}(\rho) + \frac{\alpha}{2n(1-\alpha)} \log_2(1-\varepsilon)$$

Take $n \to \infty$

$$\liminf_{n\to\infty} \frac{1}{n} QDL^{\varepsilon}(\rho^{\otimes n}) \geq H_{\frac{1}{2\alpha-1}}(\rho)$$

holds $\forall \alpha \in (1/2, 1)$

take $\alpha \to 1$ limit

$$\Rightarrow \liminf_{n\to\infty} \frac{1}{n} QDL^{\varepsilon}(\rho^{\otimes n}) \geq H(\rho)$$

$$\Rightarrow \widetilde{QDL}(\rho) \geq H(\rho)$$

von Neumann entropy