Two kinds of error:

Type I "false alarm" error probability

$$\mathrm{Tr}[(I-\Lambda)\rho]$$

Type II "missed detection" error prob.:

$$\mathrm{Tr}[\Lambda\sigma]$$

In asymmetric hypothesis testing, place a constraint on Type I error prob. + minimize Type II error prob.

For $\varepsilon\in[0,1]$,

$$\beta_\varepsilon(\rho\|\sigma) = \min_{\Lambda}\left\{\mathrm{Tr}[\Lambda\sigma] : \mathrm{Tr}[(I-\Lambda)\rho]\le\varepsilon,\ 0\le\Lambda\le I\right\}$$

error exponent or hypothesis testing relative entropy:

$$D_H^\varepsilon(\rho\|\sigma) = -\log\beta_\varepsilon(\rho\|\sigma)$$

this quantity obeys data processing

$$D_H^\varepsilon(\rho\|\sigma) \geq D_H^\varepsilon(\mathcal{N}(\rho)\|\mathcal{N}(\sigma))$$

where $\mathcal{N}$ is a q. channel

Intuition: If you perform a channel before doing a measurement to distinguish ρ from σ, then this cannot do better than performing optimal measurement.

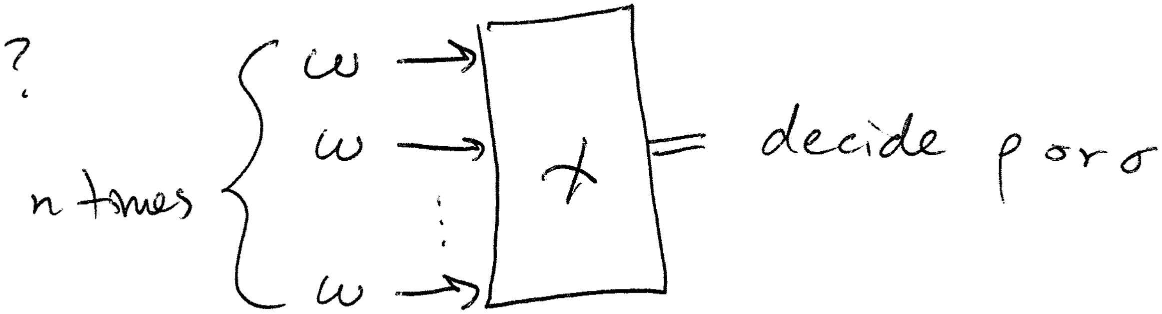
Q. Stein's lemma

$$\forall \varepsilon \in (0,1) \quad \lim_{n\to\infty} \frac{1}{n} D_H^\varepsilon(\rho^{\otimes n}\|\sigma^{\otimes n}) = D(\rho\|\sigma)$$

where $D(\rho\|\sigma) = \mathrm{Tr}[\rho(\log\rho - \log\sigma)]$

$= \langle \log\rho - \log\sigma\rangle_\rho$ is q. relative entropy

$\omega = \rho$ or σ?



optimal measurement in general is a <u>collective</u> measurement on all n systems, which is hard to implement.

How to prove Stein's lemma?

can appeal to q. Renyi relative entropies + their properties

upper bound

$$D_H^{\varepsilon}(\rho\|\sigma) \leq \widetilde{D}_\alpha(\rho\|\sigma) + \frac{\alpha}{\alpha-1}\log\left(\frac{1}{1-\varepsilon}\right) \quad \forall \varepsilon \in (0,1) \quad \forall \alpha > 1$$

where $\widetilde{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha-1}\log \operatorname{Tr}\left[\left(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\,\sigma^{\frac{1-\alpha}{2\alpha}}\right)^{\alpha}\right]$

Can show that $\lim_{\alpha\to 1} \widetilde{D}_\alpha(\rho\|\sigma) = D(\rho\|\sigma)$

& $\widetilde{D}_\alpha(\rho_1 \otimes \rho_2 \| \sigma_1 \otimes \sigma_2) = \widetilde{D}_\alpha(\rho_1\|\sigma_1) + \widetilde{D}_\alpha(\rho_2\|\sigma_2)$

(Additivity)

then $\frac{1}{n} D_H^\varepsilon(\rho^{\otimes n}\|\sigma^{\otimes n}) \leq \frac{1}{n} \widetilde{D}_\alpha(\rho^{\otimes n}\|\sigma^{\otimes n}) + \frac{\alpha}{n(\alpha-1)} \log\left(\frac{1}{1-\varepsilon}\right)$

$= \widetilde{D}_\alpha(\rho\|\sigma) + \frac{\alpha}{n(\alpha-1)} \log\left(\frac{1}{1-\varepsilon}\right)$

take $n \to \infty$ limit

$\Rightarrow \lim_{n\to\infty} \frac{1}{n} D_H^\varepsilon(\rho^{\otimes n}\|\sigma^{\otimes n}) \leq \widetilde{D}_\alpha(\rho\|\sigma) \quad \forall \alpha > 1$

take $\alpha \to 1$ limit

$\Rightarrow \lim_{n\to\infty} \frac{1}{n} D_H^\varepsilon(\rho^{\otimes n}\|\sigma^{\otimes n}) \leq D(\rho\|\sigma)$

For lower bound, use

$$D_H^{\varepsilon}(\rho\|\sigma) \geq D_\alpha(\rho\|\sigma) + \frac{\alpha}{\alpha-1}\log\left(\frac{1}{\varepsilon}\right) \qquad \forall \varepsilon \in (0,1), \ \alpha \in (0,1)$$

where $D_\alpha(\rho\|\sigma) = \frac{1}{\alpha-1}\log \mathrm{Tr}[\rho^\alpha \sigma^{1-\alpha}]$

Also have $\lim_{\alpha \to 1} D_\alpha(\rho\|\sigma) = D(\rho\|\sigma)$

+ Additivity.

Then

$$\frac{1}{n} D_H^{\varepsilon}(\rho^{\otimes n}\|\sigma^{\otimes n}) \geq \frac{1}{n} D_\alpha(\rho^{\otimes n}\|\sigma^{\otimes n}) + \frac{\alpha}{n(\alpha-1)}\log\left(\frac{1}{\varepsilon}\right)$$

$$= D_\alpha(\rho\|\sigma) + \qquad ''$$

+ take $n \to \infty$ limit followed by $\alpha \to 1$ limit