Lecture 10

Ensembles & classical-quantum states

ensemble $\{(p(x), \rho^x)\}_{x \in \mathcal{X}}$

consists of a probability distribution w/ each probability $p(x)$ paired up w/ a q. state ρ^x.

Suppose Alice prepares q. ~~state~~ system in state ρ^x w/ prob. $p(x)$. Then ~~Bob~~ she sends to Bob, who does not know which x was selected.

Bob's stated described by

$$\sum_x p(x)\, \rho^x = \rho$$

consistent w/ Born rule & probability theory

$$p(y|x) = \mathrm{Tr}[M^y \rho^x]$$

$$p(y) = \sum_x p(x)\, p(y|x) = \sum_x p(x)\, \mathrm{Tr}[M^y \rho^x]$$

$$= \mathrm{Tr}[M^y \underbrace{\sum_x p(x)\, \rho^x}_{}]$$

density operator consistent w/ measurement outcomes

ensembles are in one-to-one correspondence w/ classical-quantum states

$$\rho_{XB} = \sum_{x \in \mathcal{X}} p(x)\, |x\rangle\langle x|_X \otimes \rho^x_B$$

X is a classical register that provides a label for the state in B.

If we discard X, this amounts to partial trace, leaving

$$\mathrm{Tr}_X[\rho_{XB}] = \sum_x p(x)\, \mathrm{Tr}[|x\rangle\langle x|]\, \rho_B^x$$

$$= \sum_x p(x)\, \rho_B^x$$

consistent w/ previous discussion for ensembles.

Classical-quantum states have a block-diagonal structure

$$\rho_{XB} = \bigoplus_x p(x)\, \rho^x = \begin{bmatrix} p(x_1)\rho^{x_1} & & & \\ & p(x_2)\rho^{x_2} & & \\ & & \ddots & \\ & & & p(x_{|X|})\rho^{|X|} \end{bmatrix}$$

is a c-q state separable or entangled?

partial transpose - connected to entanglement

Action of transpose superoperator T given by

$$T(X) = \sum_{ij} |i\rangle\langle j| X |i\rangle\langle j|$$

partial transpose is an extension of this to

$$T_B \equiv (id_A \otimes T_B)(\cancel{X}_{AB})$$
$$= \sum_{ij} \left(I_A \otimes |i\rangle\langle j|_B\right)\left(X_{AB}\right)\left(I_A \otimes |i\rangle\langle j|_B\right)$$

Key observation:

partial transpose acting on
a separable state

$$(\mathrm{id}_A \otimes T_B)(\sigma_{AB})$$

$$= (\mathrm{id}_A \otimes T_B)\left(\sum_x p(x)\, \omega_A^x \otimes \tau_B^x\right)$$

$$= \sum_x p(x)\, \omega_A^x \otimes \underbrace{T}_{\text{usual transpose}}(\tau_B^x)$$

if $\tau_B^x \geq 0$ then $T(\tau_B^x) \geq 0$

$$\Rightarrow (\mathrm{id}_A \otimes T_B)(\sigma_{AB}) \geq 0$$

Thus, the partial transpose of a separable state leads to a state.

If σ_{AB} is separable, then it has a positive partial transpose.

<u>Contrapositive</u>: If σ_{AB} has a negative PT, then it is entangled.

can define the set of PPT states (positive partial transpose)

$$PPT(A{:}B) = \{\sigma_{AB} : \sigma_{AB} \geq 0,\ T_B(\sigma_{AB}) \geq 0,\ Tr[\sigma_{AB}] = 1\}$$

Measurements in QM

POVM $\{M_x\}_{x \in \mathcal{X}}$ s.t.

$$M_x \geq 0 \quad \forall x \quad \& \quad \sum_x M_x = I$$

probability of getting outcome x when performing $\{M_x\}_x$ on state ρ is

$$Pr[X=x] = Tr[M_x \rho].$$

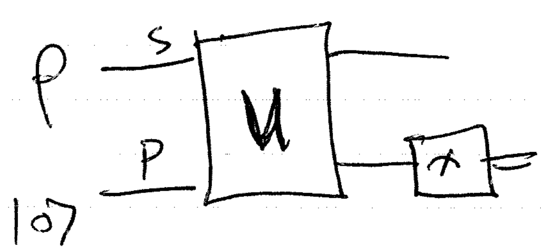
If $\{M_x\}_x$ consists of all projections, then called projective measurement

<u>Naimark's Theorem</u>: can realize an arbitrary POVM by performing unitary action between system + probe + then <u>von Neumann</u> measurement on probe.

<u>Thm</u>: For every POVM $\{M_x\}_x$, $\exists$ isometry V such that

$$M_x = V^\dagger (I \otimes |x\rangle\langle x|) V$$

<u>Interpretation</u>



system-probe interaction

$$\rho \to \rho \otimes |0\rangle\langle 0| \to U(\rho \otimes |0\rangle\langle 0|)U^\dagger$$

$$= V\rho V^{\dagger}$$

then measure & probability is

$$\Pr[X=x] = \mathrm{Tr}\left[\left(I \otimes |x\rangle\langle x|\right) V\rho V^{\dagger}\right]$$

$$= \mathrm{Tr}\left[V^{\dagger}\left(I \otimes |x\rangle\langle x|\right) V\rho\right]$$

$$= \mathrm{Tr}\left[M_x \rho\right]$$

<u>Proof of theorem:</u>

construct $V = \sum_x \sqrt{M_x} \otimes |x\rangle$

$$V^{\dagger}V = \left(\sum_{x'} \sqrt{M_{x'}} \otimes \langle x'|\right)\left(\sum_x \sqrt{M_x} \otimes |x\rangle\right)$$

$$= \sum_{x', x} \sqrt{M_{x'}}\sqrt{M_x} \langle x'|x\rangle$$

$$= \sum_x M_x = I \Rightarrow V \text{ is isometry}$$

can check that

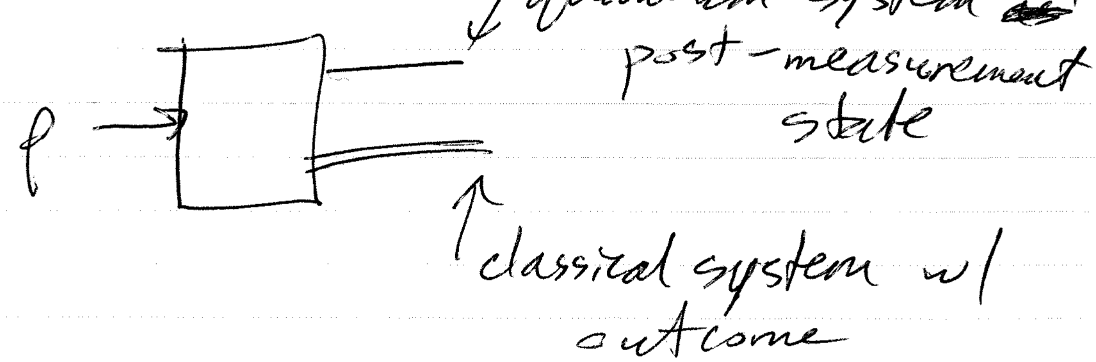
$$M_x = V^\dagger (I \otimes |x\rangle\langle x|) V$$

$$\left(\sum_{x'} \sqrt{M_{x'}} \otimes \langle x'|\right)\left(I \otimes |x\rangle\langle x|\right)\left(\sum_{x''} \sqrt{M_{x''}} \otimes |x''\rangle\right)$$

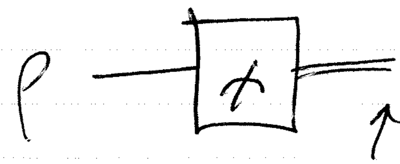
$$= \sum_{x', x''} \sqrt{M_{x'}}\sqrt{M_{x''}} \langle x'|x\rangle\langle x|x''\rangle$$

$$= M_x$$

More general picture of measurements involves post-measurement state. Can think of process as a channel



w/ POVM, the process looks like



↑ classical

& there is no q. output.

<u>quantum instrument</u> is a channel of the form

$$\mathcal{M}(\rho) = \sum_x \mathcal{M}_x(\rho) \otimes |x\rangle\langle x|$$

where each $\mathcal{M}_x$ is a CP map such that $\sum_x \mathcal{M}_x$ is TP.

probability of obtaining outcome x is $\mathrm{Tr}[\mathcal{M}_x(\rho)]$ & conditioned on this outcome the post-meas. state is

$$\frac{\mathcal{M}_x(\rho)}{\mathrm{Tr}[\mathcal{M}_x(\rho)]}$$

Since each $\mathcal{M}_x$ is CP, $\exists$ $\{K_{x,y}\}_y$ such that

$$\mathcal{M}_x(\rho) = \sum_y K_{x,y}\, \rho\, K_{x,y}^\dagger$$

& TP condition is that

$$\sum_{x,y} K_{x,y}^\dagger K_{x,y} = I$$

If we trace out quantum register, then reduced state is

$$\sum_x \mathrm{Tr}[\mathcal{M}_x(\rho)]\, |x\rangle\langle x|$$

$$= \sum_x \mathrm{Tr}\left[\sum_y K_{x,y}\,\rho K_{x,y}^\dagger\right] |x\rangle\langle x|$$

$$= \sum_x \mathrm{Tr}\left[\sum_y K_{x,y}^\dagger K_{x,y}\,\rho\right] |x\rangle\langle x|$$

consistent w/ POVM w/

elements $\left\{\sum_y K_{x,y}^\dagger K_{x,y}\right\}_x$