Lecture 8

Axioms of Quantum Mechanics

1. Quantum systems are associated w/ Hilbert spaces.
   State of a system described by a density operator ρ, satisfying $\rho \geq 0$ + $\mathrm{Tr}\{\rho\} = 1$
2. Bipartite & multipartite systems: Distinct systems A & B w/ associated Hilbert spaces $\mathcal{H}_A$ & $\mathcal{H}_B$ can be brought together to realize a composite system ~~desc~~ AB w/ associated Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$.

3. Measurement of a q. system A is described by a positive operator-valued measure $\{M_x\}_{x\in\mathcal{X}}$ s.t. $M_x \geq 0 \quad \forall x \in \mathcal{X}$ +

$$\sum_{x\in\mathcal{X}} M_x = I$$

If state of system is ρ, then probability of obtaining outcome x is

$$\Pr[x] = \mathrm{Tr}[M_x \rho]$$

A physical observable O corresponds to a Hermitian operator, w/ spectral decomposition

$$O = \sum_{\lambda\in \mathrm{spec}(O)} \lambda \, \Pi_\lambda$$

where $\mathrm{spec}(O)$ is the set of distinct eigenvalues of O & Π_λ is a spectral projection.

Measurement of observable O is described by POVM $\{\Pi_\lambda\}_\lambda$, indexed by eigenvalues of O.

Expected value of obs. O for state ρ is

$$\langle O \rangle_\rho = \mathrm{Tr}[O\rho] = \sum_{\lambda \in \mathrm{spec}(O)} \lambda \,\mathrm{Tr}[\Pi_\lambda \rho]$$

4. Evolution of state of q. system described by q. channel, which is a linear, completely positive, trace-preserving map.
(includes unitary evolution as special case)

Most general q. system we consider is qudit (d-dimensional state). Hilbert space spanned by $\{|0\rangle, |1\rangle, \ldots, |d-1\rangle\}$.

general qudit density operator can be expanded in terms of this basis as

$$\rho = \sum_{i,j=0}^{d-1} \rho_{ij} \, |i\rangle\langle j|$$

density operators form a convex set:

if $\{\rho^x\}_{x \in \mathcal{X}}$ is a set of states, then $\sum_{x \in \mathcal{X}} p(x) \, \rho^x$ is a state, where $\{p(x)\}_x$ is a prob. dist.

extreme points in convex set of states are <u>pure</u> states, i.e. unit rank projections onto unit vectors. They have the form $|\psi\rangle\langle\psi|$ for $|\psi\rangle \in \mathcal{H}$.

often call $|\psi\rangle$ a <u>state vector</u>

every density operator ρ can be written as a convex combo of pure states:

$$\rho = \sum_x p(x)\, |\psi_x\rangle\langle\psi_x| .$$

State that are not pure are called <u>mixed states</u>.

state is <u>maximally mixed</u> if $\rho = \frac{I}{d}$.

Bloch sphere: (for qubits, & see book for extension to qudits)

$$\rho = \frac{1}{2}(I + r_1 X + r_2 Y + r_3 Z)$$

where $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$,

Pauli matrices

$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

trace is always equal to one b/c Paulis are traceless.

ρ is positive semidefinite if & only if $\|\vec{r}\|_2 = \sqrt{r_1^2 + r_2^2 + r_3^2} \leq 1$

can think of $\vec{r}$ as coordinates in a unit sphere.

if $\|\vec{r}\|_2 = 1$ then pure state

if $\|\vec{r}\|_2 < 1$ then mixed state

Bipartite States + Schmidt decomposition

can write an arbitrary bipartite ~~state~~ vector as

$$|\psi\rangle_{AB} = \sum_{ij} \alpha_{ij} \, |i\rangle_A |j\rangle_B$$

where $\|\vec{\alpha}\|_2 = 1$,

From Schmidt decomposition, we have

$$|\psi\rangle_{AB} = \sum_{k=1}^{r} \sqrt{\lambda_k} \, |e_k\rangle_A \otimes |f_k\rangle_B$$

r is Schmidt rank

if $r=1$, then state vector is <u>product</u>

if $r>1$, then state vector is <u>entangled</u>.

partial trace

usual trace is $Tr[X] = \sum_i \langle i|X|i\rangle$

partial trace generalizes this

Given X_{AB}, partial trace over B is

$$X_A \equiv Tr_B [X_{AB}]$$
$$= (id_A \otimes Tr)[X_{AB}]$$
$$= \sum_j (I_A \otimes \langle j|_B)\, X_{AB} \,(I_A \otimes |j\rangle_B)$$

What does $I_A \otimes \langle j|$ mean?

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \otimes [0 \cdots 0\, 1\, 0 \cdots 0]$$

$d_A \times d_A d_B$ matrix $\qquad X_{AB}$ is $d_A d_B \times d_A d_B$

$I_A \otimes |j\rangle$ is

$$\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad d_A d_B \times d_A$$

So $\mathrm{Tr}_B\{X_{AB}\}$ is a $d_A \times d_A$ matrix

---

other way of thinking about partial trace:

$$\mathrm{Tr}\left[(M_A \otimes I_B)\,\rho_{AB}\right] = \mathrm{Tr}\left[M_A \underset{\uparrow \atop \mathrm{Tr}_B[\rho_{AB}]}{\rho_A}\right]$$

to figure out expectations of local observables, we only need reduced state (global picture consistent w/ local picture)

Also,

$$\mathrm{Tr}_B\left[|i\rangle\langle j|_A \otimes |k\rangle\langle l|_B\right]$$

$$= |i\rangle\langle j|_A \;\delta_{k,l}$$

$$= |i\rangle\langle j|_A \;\langle l|k\rangle$$

<u>Consequence</u>: ~~the~~ marginal states of every pure bipartite state $|\psi\rangle\langle\psi|_{AB}$ have same non-zero eigenvalues

<u>Proof</u>: use Schmidt

$$\mathrm{Tr}_B\left[|\psi\rangle\langle\psi|_{AB}\right]$$

$$= \mathrm{Tr}_B\left[\sum_{k,k'=1}^{r} \sqrt{\lambda_k \lambda_{k'}}\; |e_k\rangle\langle e_{k'}| \otimes |f_k\rangle\langle f_{k'}|\right]$$

$$= \sum_{k,k'=1}^{r} \sqrt{\lambda_k \lambda_{k'}}\; \mathrm{Tr}_B\left[|e_k\rangle\langle e_{k'}| \otimes |f_k\rangle\langle f_{k'}|\right]$$

$$= \sum_{k,k'=1}^{r} \sqrt{\lambda_k \lambda_{k'}} \, |e_k\rangle\langle e_{k'}| \, \delta_{k,k'}$$

$$= \sum_{k=1}^{r} \lambda_k \, |e_k\rangle\langle e_k|$$

similar calculation gives

$$\mathrm{Tr}_A\left[|\psi\rangle\langle\psi|_{AB}\right] = \sum_{k=1}^{r} \lambda_k \, |f_k\rangle\langle f_k|$$

Separable & Entangled States

Bipartite state σ_{AB} is separable

if $\exists$ prob. dist. $p(x)$, $\{\omega_A^x\}_x$ & $\{\tau_B^x\}_x$ states such that

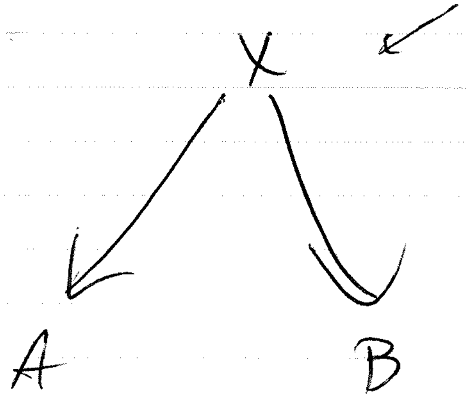
$$\sigma_{AB} = \sum_x p(x) \, \omega_A^x \otimes \tau_B^x$$

i.e. convex combination of product states

states that are not separable are entangled.

deciding separability is a hard computational problem.

separable states can be prepared by local operations & classical communication (LOCC)



classical X is chosen, copied, randomly & sent to A & B, who prepare ω^X & τ^X, resp.

like a Markov process in the background.

A pure state $|\psi\rangle\langle\psi|_{AB}$ is

<u>maximally entangled</u> if

the Schmidt coefficients are all equal to $\frac{1}{\sqrt{d}}$, where d is Schmidt rank.

i.e. if

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{d}} \sum_{k=1}^{d} |e_k\rangle_A \otimes |f_k\rangle_B$$

<u>Bell states</u>

$$|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|00\rangle_{AB} + |11\rangle_{AB}\right)$$

$$|\Phi^-\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|00\rangle_{AB} - |11\rangle_{AB}\right)$$

$$|\Psi^+\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|01\rangle_{AB} + |10\rangle_{AB}\right)$$

$$|\Psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle_{AB}\right)$$

Let us stress the difference between <u>coherent superposition</u> & <u>probabilistic</u> or <u>convex mixture</u>

the state vector $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ corresponds to density operator

$$|\psi\rangle\langle\psi| = \left(\alpha|0\rangle + \beta|1\rangle\right)\left(\alpha^*\langle 0| + \beta^*\langle 1|\right)$$

$$= |\alpha|^2 |0\rangle\langle 0| + \beta\alpha^* |1\rangle\langle 0|$$

$$+ \alpha\beta^* |0\rangle\langle 1| + |\beta|^2 |1\rangle\langle 1|$$

$$= \begin{bmatrix} |\alpha|^2 & \beta\alpha^* \\ \alpha\beta^* & |\beta|^2 \end{bmatrix}$$

off diagonal terms are present

Now consider a convex mixture of $|0\rangle\langle 0|$ & $|1\rangle\langle 1|$

$$|\alpha|^2 \, |0\rangle\langle 0| + |\beta|^2 \, |1\rangle\langle 1|$$

$$= \begin{bmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{bmatrix}$$

<u>no</u> off diagonal terms!

The difference between a coherent superposition & a probabilistic mixture is physically observable via a <u>measurement</u>.

Which observable will show a difference? Z? X?

(Paulis)