$\boxed{\text{Lecture 5}}$

continuing w/ Chap. 2

functions of Hermitian operators

Let $f: \mathbb{R} \to \mathbb{R}$ be a real-valued function w/ domain $\mathrm{dom}(f)$

For Hermitian $X = \sum_k \lambda_k |\psi_k\rangle\langle\psi_k|$ we define

$$f(X) = \sum_{k:\, \lambda_k \in \mathrm{dom}(f)} f(\lambda_k) |\psi_k\rangle\langle\psi_k|$$

consistent w/ Taylor expansion for analytic functions

suppose $f(z) = \sum_{\ell=0}^{\infty} \alpha_z z^\ell$

Then given $X = U D U^\dagger$

diagonal

$$X^\ell = (U D U^\dagger)^\ell = U D^\ell U^\dagger$$

$$\Rightarrow f(X) = \sum_{\ell=0}^{\infty} \alpha_\ell X^\ell$$

$$= \sum_{\ell=0}^{\infty} \alpha_\ell U D^\ell U^\dagger$$

$$= U \left(\sum_{\ell=0}^{\infty} \alpha_\ell D^\ell \right) U^\dagger$$

$$= U f(D) U^\dagger$$

function applied to diagonal entries individually

$$= \sum_k f(\lambda_k) |\psi_k\rangle\langle\psi_k|$$

Consequence:

if U is unitary, then

$$f(UXU^{\dagger}) = U f(X) U^{\dagger}$$

Examples of functions: x^{α}, $\log x$
(used throughout, for Renyi & von Neumann entropies)

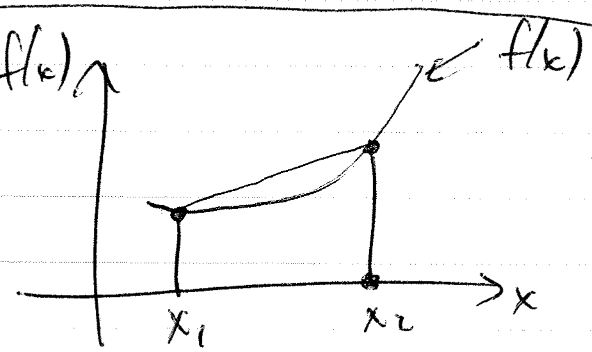
<u>operator convex, concave, & monotone functions</u>

f is <u>operator convex</u> if

$$f(\lambda X + (1-\lambda) Y) \leq \lambda f(X) + (1-\lambda) f(Y)$$

$\forall$ Hermitian X, Y & $\lambda \in [0,1]$

graph of convex function



function always lies below chord connecting two points

(4)

operator convexity is stronger than ordinary convexity

<u>examples of operator convex functions:</u>

x^{α} for $\alpha \in [-1,0) \cup [1,2]$ & $x \in [0,\infty)$

$x \log x$ for $x \in [0,\infty)$

f is operator concave if
$-f$ is operator convex.

f is operator monotone if
$$X \leq Y \Rightarrow f(X) \leq f(Y)$$
for all Hermitian X & Y

f is operator anti-monotone if
$-f$ is operator monotone

these concepts are used in the proofs of data-processing inequalities for q. relative entropies.

Norms

norm is a function $\|\cdot\|: \mathcal{H} \to \mathbb{R}$ that satisfies

1) Non-negativity:

$$\| |\psi\rangle \| \geq 0 \quad \forall\ |\psi\rangle \in \mathcal{H} \ \&$$

$$\| |\psi\rangle \| = 0 \quad \text{iff} \quad |\psi\rangle = 0$$

2) Scaling / homogeneity

For all $\alpha \in \mathbb{C}$ & $|\psi\rangle \in \mathcal{H}$,

$$\| \alpha |\psi\rangle \| = |\alpha| \, \| |\psi\rangle \|$$

3) Triangle inequality:

$\forall\ |\psi\rangle, |\phi\rangle \in \mathcal{H}$

$$\big\|\, |\psi\rangle + |\phi\rangle \,\big\| \leq \big\|\, |\psi\rangle \,\big\| + \big\|\, |\phi\rangle \,\big\|$$

Consequence of 2) + 3):
every norm is convex

$$\big\|\, \lambda|\psi\rangle + (1-\lambda)|\phi\rangle \,\big\| \leq \lambda \big\|\, |\psi\rangle \,\big\| + (1-\lambda)\big\|\, |\phi\rangle \,\big\|$$

$\forall\ |\psi\rangle, |\phi\rangle \in \mathcal{H}$ $\forall\ \lambda \in [0,1]$

Norms for the Hilbert space of linear operators:

For all $X \in L(\mathcal{H})$, Schatten α-norm

$$\|X\|_\alpha = \left(\mathrm{Tr}\left[|X|^\alpha\right]\right)^{1/\alpha}$$

$\forall\ \alpha \in [1, \infty)$

where $|X| = \sqrt{X^+ X}$

$$\|X\|_\infty = \lim_{\alpha \to \infty} \|X\|_\alpha$$

can show that

$$\|X\|_\alpha = \left(\sum_{k=1}^{r} s_k^\alpha \right)^{1/\alpha}$$

where $\{s_k\}_{k=1}^{r}$ is set of singular values of X.

Basic Properties of Schatten Norm

<u>Monotonicity:</u> $\alpha \geq \beta$

$$\Rightarrow \|X\|_\alpha \leq \|X\|_\beta$$

monotonically non-increasing w/ α

<u>Isometric invariance:</u>

$\|X\|_\alpha = \|UXV^+\|_\alpha$ $\forall$ isometries U & V

Duality:

$$\|X\|_\alpha = \sup_{Y \neq 0} \left\{ \frac{|\mathrm{Tr}[Y^\dagger X]|}{\|Y\|_\beta}, \ \frac{1}{\alpha} + \frac{1}{\beta} = 1 \right\}$$

$\Rightarrow$ Hölder's inequality:

$$|\mathrm{Tr}[Z^\dagger X]| \leq \|X\|_\alpha \|Z\|_\beta$$

where $\alpha, \beta \in [1, \infty]$ satisfy

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1$$

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can prove that

$$\|X\|_\infty = s_{max}$$

largest singular value of X

Trace Norm $(\alpha = 1)$

$$\|X\|_1 = \sum_{k=1}^{r} s_k$$

sum of singular values

If X is Hermitian, then

$$\|X\|_1 = \sum_k |\lambda_k|$$

eigenvalues

If X is positive semidefinite, then

$$\|X\|_1 = \mathrm{Tr}[X]$$

Variational characterization of trace norm

$$\|X\|_1 = \sup_{Y \neq 0,\ \|Y\|_\infty \leq 1} \left| \mathrm{Tr}[Y^\dagger X] \right|$$

Also, $\|X\|_1 = \sup_{U} \left| \mathrm{Tr}[U X] \right|$

isometry

we have

$$\|X\|_\infty \leq \|X\|_1 \leq d \|X\|_\infty$$

Operator Jensen Inequality

Let f be a continuous function. The following are equivalent:

1) f is operator convex

2) $$f\left(\sum_k A_k^+ X_k A_k\right) \leq \sum_k A_k^+ f(X_k) A_k$$

$\forall$ Hermitian X_k & $\{A_k\}_k$ satisfying $\sum_k A_k^+ A_k = I$

3) $$f(V^+ X V) \leq V^+ f(X) V$$

$\forall$ Hermitian X & isometry V.

$2 \Rightarrow 1$ easy. pick

$$A_1 = \sqrt{r}\, I \,,\ X_1 = X \,,$$

$$A_2 = \sqrt{1-r}\, I \,,\ X_2 = Y$$

$$\Rightarrow f(rX + (1-r)Y) \leq r f(X) + (1-r) f(Y)$$

$3 \Rightarrow 2$ more interesting

Set $X = \sum_k X_k \otimes |k\rangle\langle k|$

$$V = \sum_k A_k \otimes |k\rangle$$

$V^+ V = I$ follows from $\sum_k A_k^+ A_k = I$

Also $V^+ X V = \sum_k A_k^+ X_k A_k$

$$f(X) = f\left(\sum_k X_k \otimes |k\rangle\langle k|\right) = \sum_k f(X_k) \otimes |k\rangle\langle k|$$

Then $V^\dagger f(X) V = \sum_k A_k^\dagger f(X_k) A_k$

for others, see book

Superoperators

linear operation that acts on linear operators.

$$\mathcal{N} : L(\mathcal{H}_A) \to L(\mathcal{H}_B)$$

satisfies

$$\mathcal{N}(\alpha X + \beta Y) = \alpha \mathcal{N}(X) + \beta \mathcal{N}(Y)$$

$$\forall \alpha, \beta \in \mathbb{C} \ \& \ X, Y \in L(\mathcal{H}_A).$$

For every superoperator, $\exists\, n \in \mathbb{N}$ & sets $\{K_i\}_{i=1}^n$ & $\{L_i\}_{i=1}^n$ s.t.

$$\mathcal{N}(X) = \sum_{i=1}^n K_i X L_i^\dagger$$

identity denoted by

id & satisfies

$$\mathrm{id}(X) = X \quad \forall X \in L(\mathcal{H})$$

Given superoperators N & M,

their tensor product is given by

$N \otimes M$ & satisfies

$$(N \otimes M)(X \otimes Y) = N(X) \otimes M(Y)$$

Hermiticity preserving superoperator

N is H.P. if

$N(X)$ is Hermitian $\forall$ Hermitian X.

Adjoint of a superoperator

Adjoint $\mathcal{N}^\dagger$ of $\mathcal{N}$ is unique superoperator satisfying

$$\langle Y, \mathcal{N}(X)\rangle = \langle \mathcal{N}^\dagger(Y), X\rangle$$

$$\forall X, Y$$

can show that

$$\mathcal{N}^\dagger(Y) = \sum_{i=1}^{n} K_i^\dagger Y L_i$$

if $\mathcal{N}(X) = \sum_{i=1}^{n} K_i X L_i^\dagger$

$\mathcal{N}$ is trace preserving if

$$\mathrm{Tr}[\mathcal{N}(X)] = \mathrm{Tr}[X] \quad \forall X \in \mathcal{L}(\mathcal{H})$$

$\mathcal{N}$ is unital if $\mathcal{N}(I_A) = I_B$

can show that

$\mathcal{N}$ is trace preserving iff

$\mathcal{N}^{\dagger}$ is unital.

Analysis

meaning of $\lim_{n\to\infty} s_n = \ell$:

$\forall \varepsilon > 0 \;\; \exists\, n_\varepsilon \in \mathbb{N}$ such that

$\forall n \geq n_\varepsilon , \quad |s_n - \ell| < \varepsilon$

Infimum

Let $E \subset \mathbb{R}$

x is a lower bound for E

if $y \geq x \;\; \forall y \in E$.

Infimum is the greatest lower bound.

<u>Example:</u> $E = \{\frac{1}{n}\}_{n \in \mathbb{N}}$

$\inf E = 0$ but $0 \notin E$.

<u>supremum</u>

x is an upper bound for E if $y \leq x$.

supremum is the least upper bound.

we often use the following common abbreviations:

$$\inf_{X \in S} F(X) = \inf\{F(X) : X \in S\}$$

where S is a subset of $L(H)$

$$\sup_{X \in S} F(X) = \sup\{F(X) : X \in S\}$$

limit of a sequence need not exist

e.g. $+1, -1, +1, -1, \ldots$

nice to have a substitute:

lim inf & lim sup

<u>liminf</u>

s is an <u>asymptotic lower bound</u> for sequence $\{s_n\}_{n\in\mathbb{N}}$
if $\forall \varepsilon > 0 \ \exists n_\varepsilon \in \mathbb{N}$ s.t.

$$\forall n \geq n_\varepsilon \quad s_n > s - \varepsilon$$

<u>limit inferior</u> is greatest asymptotic lower bound.

$$\liminf_{n\to\infty} s_n$$

lim sup

s is an asymptotic upper bound for sequence $\{s_n\}_{n \in \mathbb{N}}$ if $\forall \varepsilon > 0$, $\exists n_\varepsilon \in \mathbb{N}$ s.t. $\forall n \geq n_\varepsilon \quad s_n < s + \varepsilon$

limit superior is least asymptotic upper bound

$$\limsup_{n \to \infty} s_n$$

we always have

$$\liminf_{n \to \infty} s_n \leq \limsup_{n \to \infty} s_n$$

If opposite inequality holds, then

$$\liminf_{n \to \infty} s_n = \limsup_{n \to \infty} s_n = \lim_{n \to \infty} s_n$$

Compact sets

subset S is <u>compact</u> if every sequence of elements in S has a subsequence that converges to an element in S.

For compact sets, infimum & supremum always exist and are contained in the set.

So

$$\inf_{x \in S} F(x) = \min_{x \in S} F(x) \quad \&$$

$$\sup_{x \in S} F(x) = \max_{x \in S} F(x)$$

for compact S & continuous F.

Convex set

subset C is convex if

$\forall u, v \in C$ & $\lambda \in [0,1]$

$$\lambda u + (1-\lambda) v \in C$$

Given a set $\{v_x\}_x$,

$\sum_x p(x) v_x$ is called a

convex combination of $\{v_x\}$

where $p(x)$ is a prob. dist.

convex hull is the convex set of all convex combinations of vectors in a set.

Minimax theorem

Suppose we have

$$\inf_{x \in S} \sup_{y \in S'} F(x,y)$$

can think of this in

terms of a two-player game & a payoff / cost function.

we always have

$$\sup_{y \in S'} \inf_{x \in S} F(x,y) \leq \inf_{x \in S} \sup_{y \in S'} F(x,y)$$

going second in a game gives an advantage.

When does the reverse inequality hold?

<u>minimax theorem</u> (simple version)

If S & S' are compact & convex & F is continuous, convex in X for fixed Y, concave in Y for fixed X, then

$$\sup_{Y \in S'} \inf_{X \in S} F(X,Y) = \inf_{X \in S} \sup_{Y \in S'} F(X,Y)$$

$\Rightarrow$ no benefit to playing second. Both players just employ optimal strategies