PHYS 7895: Gaussian Quantum Information

Lecture 10

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1 Overview

In the last lecture, we represented faithful Gaussian states as thermal states of quadratic Hamiltonians and discussed the Williamson theorem.

In this lecture, we first review a method to find the symplectic eigenvalues of a positive definite matrix. We then derive a relation between the Hamiltonian matrix and the covariance matrix corresponding to a faithful Gaussian state. Finally, we determine formulas for the purity and von Neumann entropy of a Gaussian state. We point readers to [Ser17] for background on some of the topics covered in this lecture.

2 Symplectic eigenvalues of a positive definite matrix

In this section, we discuss a method to find the symplectic eigenvalues of a positive definite matrix M.

Let Ω denote the real, canonical, anti-symmetric form defined as

$$\Omega = I_n \otimes \Omega_1, \tag{1}$$

where

$$\Omega_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{2}$$

which encodes the canonical commutation relations of the quadrature operators. Note that $\Omega\Omega^T = -\Omega^2 = I$.

Let S denote a sympletic matrix such that $S\Omega S^T = \Omega$. It follows that such a matrix S is invertible with inverse given by $S^{-1} = \Omega S^T \Omega^T$. We begin by showing that $\Omega S = S^{-T}\Omega$. This is a direct consequence of the fact that S^T is symplectic, which can be seen from the following steps:

$$S\Omega S^T = \Omega, \tag{3}$$

$$\Rightarrow S\Omega S^T \Omega = -I, \tag{4}$$

$$\Rightarrow S\Omega S^T \Omega S = -S, \tag{5}$$

$$\Rightarrow S^{-1} S\Omega S^T \Omega S = -S^{-1}S, \tag{6}$$

$$\Rightarrow \Omega S^T \Omega S = -I, \tag{7}$$

$$\Rightarrow S^T \Omega S = \Omega \ . \tag{8}$$

It then follows that

$$\Omega S = S^{-T}\Omega. \tag{9}$$

As discussed in the previous lecture, a positive definite $2n \times 2n$ matrix M has the following symplectic decomposition:

$$M = SDS^T, \tag{10}$$

where $d_j > 0$ for all $j \in \{1, \dots, n\}$,

$$D = \bigoplus_{j=1}^{n} d_j \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = D_n \otimes I_2, \tag{11}$$

and

$$D_n = \operatorname{diag}(d_1, d_2, \dots, d_n). \tag{12}$$

We now establish a connection between the symplectic eigenvalues of a positive definite matrix M and the eigenvalues of the matrix $i\Omega M$. Consider the following chain of equalities:

$$\begin{aligned}
i\Omega M &= i\Omega SDS^T && (13)\\
&= i\Omega S(D_n \otimes I_2)S^T && (14)\\
&= S^{-T}(i\Omega)(D_n \otimes I_2)S^T && (15)\\
&= S^{-T}(I_n \otimes i\Omega_1)(D_n \otimes I_2)S^T && (16)\\
&= S^{-T}(D_n \otimes i\Omega_1)S^T && (17)\\
&= S^{-T}(D_n \otimes -\sigma_Y)S^T && (18)\\
&= \underbrace{S^{-T}(I_n \otimes U_2)}_{B}(D_n \otimes -\sigma_Z)\underbrace{(I_n \otimes U_2^\dagger)S^T}_{B^{-1}}. && (19)
\end{aligned}$$

The first equality follows from (10). The second equality follows from (11). The third equality follows from (9). The fourth equality follows from the definition of Ω as defined in (1). The last two equalities follow from the fact that

$$i\Omega_1 = -\sigma_Y = U_2(-\sigma_Z)U_2^\dagger, \tag{20}$$

where

$$U_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}. \tag{21}$$

From (19) and from the fact that $D_n \otimes -\sigma_Z = \operatorname{diag}(-d_1, d_1, -d_2, d_2, \dots, -d_n, d_n)$, it follows that the usual eigendecomposition of $i\Omega M$ is given by $B(D_n \otimes -\sigma_Z)B^{-1}$, where

$$B = S^{-T}(I_n \otimes U_2) \tag{22}$$

is the matrix of eigenvectors. We note that S^{-T} can be expressed in terms of Ω and S. Since $S\Omega S^T = \Omega$, it follows that $S\Omega S^T\Omega^T = \Omega\Omega^T$. Since $\Omega\Omega^T = I$, $S^{-1} = \Omega S^T \Omega^T$. Therefore, $S^{-T} = \Omega S \Omega^T$.

Therefore, a method to find the symplectic eigenvalues of a positive definite matrix M is as follows. We first find the usual eigendecomposition of the matrix $i\Omega M$, and the corresponding eigenvalues provide the information of symplectic eigenvalues of the matrix M. Moreover, the symplectic matrix S corresponding to the transformation $M = SDS^T$, can be found from the eigenvector matrix $B = S^{-T}(I_n \otimes U_2) = \Omega S \Omega^T (I_n \otimes U_2)$ as defined in (22), i.e., $S = B^{-T}(I_n \otimes U_2^T) = \Omega^T B (I_n \otimes U_2^\dagger)\Omega$.

3 Relationship between the Hamiltonian matrix and the covariance matrix for a faithful Gaussian state

In this section, we derive the following relations between the Hamiltonian matrix and the covariance matrix corresponding to a faithful Gaussian state:

$$\sigma = \coth\left(\frac{i\Omega H}{2}\right) i\Omega, \tag{23}$$

$$H = 2\,\mathrm{arccoth}(i\Omega\sigma) i\Omega \ . \tag{24}$$

As discussed in the previous lecture, a positive definite matrix H can be represented in the following symplectic diagonalized form:

$$H = S^T \bigoplus_{j=1}^{n} \lambda_j \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} S \ , \tag{25}$$

where $\lambda_j > 0, \forall j \in \{1, \ldots, n\}$.

Moreover, the corresponding covariance matrix σ can be written as

$$\sigma = S^{-1} \bigoplus_{j=1}^{n} \coth\left(\frac{\lambda_j}{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} S^{-T} \ , \tag{26}$$

where $\nu_j \equiv \coth(\lambda_j/2)$ for $j \in \{1, \ldots, n\}$ are the symplectic eigenvalues of σ.

From (19) and (25), it follows that

$$\frac{1}{2} i\Omega H = \frac{1}{2} S^{-1}(I_n \otimes U_2)(D_n \otimes -\sigma_Z)(I_n \otimes U_2^\dagger)S, \tag{27}$$

where $D_n = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Consider the following chain of equalities:

$$\begin{aligned} \coth\left(\frac{i\Omega H}{2}\right) &= S^{-1}(I_n \otimes U_2) \coth\left(\frac{D_n \otimes -\sigma_Z}{2}\right)(I_n \otimes U_2^\dagger)S && (28)\\ &= S^{-1}(I_n \otimes U_2)\big(\coth(D_n/2) \otimes -\sigma_Z\big)(I_n \otimes U_2^\dagger)S && (29)\\ &= S^{-1}(\coth(D_n/2) \otimes i\Omega_1)S && (30)\\ &= S^{-1}(\coth(D_n/2) \otimes I_2)(I_n \otimes i\Omega_1)S && (31)\\ &= S^{-1}(\coth(D_n/2) \otimes I_2) i\Omega S && (32)\\ &= S^{-1}(\coth(D_n/2) \otimes I_2) S^{-T} i\Omega && (33)\\ &= \sigma i\Omega \ . && (34) \end{aligned}$$

The first equality follows from (27). The second equality follows from the fact that $\coth(\cdot)$ is an odd function. The third equality follows from (20). The fifth equality follows from (1). The sixth equality follows from (9). The last equality follows from (26).

Therefore, we get

$$\coth\left(\frac{i\Omega H}{2}\right) i\Omega = \sigma(i\Omega)(i\Omega) \tag{35}$$

$$= \sigma. \tag{36}$$

Similarly, the relation in (24) can be derived.

4 Uncertainty relation and symplectic eigenvalues of a covariance matrix

Previously, we proved that the following uncertainty relation holds for any n-mode quantum state that has a finite covariance matrix σ:

$$\sigma + i\Omega \geq 0 \; . \tag{37}$$

We now discuss the restriction imposed by the uncertainty relation in (37) on the symplectic eigenvalues of σ. Let S be the symplectic matrix diagonalizing σ as

$$S\sigma S^T = D = \bigoplus_{j=1}^{n} d_j \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} . \tag{38}$$

We now prove that (37) implies $d_j \geq 1, \forall j$. Consider the following chain of inequalities:

$$\sigma + i\Omega \geq 0 \tag{39}$$

$$\Rightarrow S(\sigma + i\Omega)S^T \geq 0 \tag{40}$$

$$\Rightarrow S\sigma S^T + iS\Omega S^T \geq 0 \tag{41}$$

$$\Rightarrow D + i\Omega \geq 0 \tag{42}$$

$$\Rightarrow \bigoplus_{j=1}^{n} d_j \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + i \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \geq 0 \tag{43}$$

$$\Rightarrow \bigoplus_{j=1}^{n} \begin{bmatrix} d_j & i \\ -i & d_j \end{bmatrix} \geq 0 \tag{44}$$

$$\Rightarrow \begin{bmatrix} d_j & i \\ -i & d_j \end{bmatrix} \geq 0, \forall j. \tag{45}$$

Since the eigenvalues of $\begin{bmatrix} d_j & i \\ -i & d_j \end{bmatrix}$ are $d_j + 1$ and $d_j - 1$, it follows from (45) that $d_j \geq 1, \forall j$.

Thus, any quantum covariance matrix σ (i.e., obeying (37)) has all of its symplectic eigenvalues greater than or equal to one.

5 Purification of a Gaussian state

In this section, we study Gaussian purifications of Gaussian states. We begin by determining the mean vector and covariance matrix for a tensor product of two Gaussian states.

5.1 Tensor product of two Gaussian states

Let $\bar{r}_A$ denote the mean vector and σ_A denote the covariance matrix of a Gaussian state ρ_A. Let $\bar{r}_B$ denote the mean vector and σ_B denote the covariance matrix of a Gaussian state ρ_B. Then the mean vector of the tensor product state $\rho_A \otimes \rho_B$ is given by

$$\bar{r}_{AB} \equiv \begin{bmatrix} \bar{r}_A \\ \bar{r}_B \end{bmatrix}. \tag{46}$$

Moreover, the covariance matrix of $\rho_A \otimes \rho_B$ is given by

$$\sigma_{AB} \equiv \sigma_A \oplus \sigma_B = \begin{bmatrix} \sigma_A & 0 \\ 0 & \sigma_B \end{bmatrix}. \tag{47}$$

Similarly, if the mean vector of a Gaussian state is $\begin{bmatrix} \bar{r}_A \\ \bar{r}_B \end{bmatrix}$ and the covariance matrix is $\begin{bmatrix} \sigma_A & 0 \\ 0 & \sigma_B \end{bmatrix}$, then the Gaussian state is a tensor product of two Gaussian states.

5.2 Gaussian purifications of Gaussian states

A thermal state with mean number of photons $\bar{n} \geq 0$ can be expressed in the photon-number basis as follows.

$$\theta(\bar{n}) = \frac{1}{\bar{n}+1} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{\bar{n}+1} \right)^n |n\rangle\langle n| \,. \tag{48}$$

Alternatively,

$$\theta(\lambda) = \frac{1}{z(\lambda)} \sum_{n=0}^{\infty} \exp(-\lambda(n+1/2))|n\rangle\langle n|, \tag{49}$$

where $z(\lambda) = (e^{\lambda/2} - e^{-\lambda/2})^{-1}$ for $\lambda > 0$ (note that $\lambda = \ln(1+1/\bar{n})$).

A purification of the thermal state $\theta_A(\bar{n})$ is given by the following two-mode squeezed vacuum (TMS) state:

$$|\psi_{\mathrm{TMS}}(\bar{n})\rangle_{AR} = \frac{1}{\sqrt{\bar{n}+1}} \sum_{n=0}^{\infty} \sqrt{\left(\frac{\bar{n}}{\bar{n}+1} \right)^n} |n\rangle_A |n\rangle_R, \tag{50}$$

where R is a reference system.

The covariance matrix of the two-mode squeezed vacuum state $|\psi_{\text{TMS}}(\bar{n})\rangle_{AR}$ is given by

$$\begin{bmatrix} 2\bar{n}+1 & 0 & 2\sqrt{\bar{n}(\bar{n}+1)} & 0 \\ 0 & 2\bar{n}+1 & 0 & -2\sqrt{\bar{n}(\bar{n}+1)} \\ 2\sqrt{\bar{n}(\bar{n}+1)} & 0 & 2\bar{n}+1 & 0 \\ 0 & -2\sqrt{\bar{n}(\bar{n}+1)} & 0 & 2\bar{n}+1 \end{bmatrix}, \tag{51}$$

which can be written in the following compact form:

$$\begin{bmatrix} (2\bar{n}+1)I & 2\sqrt{\bar{n}(\bar{n}+1)}\sigma_Z \\ 2\sqrt{\bar{n}(\bar{n}+1)}\sigma_Z & (2\bar{n}+1)I \end{bmatrix}. \tag{52}$$

By the Williamson theorem, any n-mode Gaussian state ρ can be written as

$$\rho = \hat{D}_{-\bar{r}}\hat{S}\left[\bigotimes_{j=1}^{n}\theta_{A_j}(\bar{n}_j)\right]\hat{S}^\dagger\hat{D}_{\bar{r}}, \tag{53}$$

where $\hat{S}$ is a unitary generated by a quadratic Hamiltonian. Then a Gaussian purification of ρ is given by

$$\left[\hat{D}_{-\bar{r}}\hat{S}\right]_{A^n}\bigotimes_{j=1}^{n}|\psi_{\text{TMS}}(\bar{n}_j)\rangle_{A_jR_j}\ . \tag{54}$$

The mean vector of this purification is $\begin{bmatrix}\bar{r}\\0\end{bmatrix}$. Moreover, the covariance matrix of this purification is

$$\begin{bmatrix} \sigma & S\bigoplus_{j=1}^{n}2\sqrt{\bar{n}_j(\bar{n}_j+1)}\sigma_Z \\ \left(\bigoplus_{j=1}^{n}2\sqrt{\bar{n}_j(\bar{n}_j+1)}\sigma_Z\right)S^T & \bigoplus_{j=1}^{n}(2\bar{n}_j+1)I_2 \end{bmatrix}. \tag{55}$$

One can arrive at this conclusion from the fact that

$$\sigma = S\left(\bigoplus_{j=1}^{n}(2\bar{n}_j+1)I_2\right)S^T \tag{56}$$

and the covariance matrix for $\bigotimes_{j=1}^{n}|\psi_{\text{TMS}}(\bar{n}_j)\rangle_{A_jR_j}$ is

$$\begin{bmatrix} \bigoplus_{j=1}^{n}(2\bar{n}_j+1)I_2 & \bigoplus_{j=1}^{n}2\sqrt{\bar{n}_j(\bar{n}_j+1)}\sigma_Z \\ \bigoplus_{j=1}^{n}2\sqrt{\bar{n}_j(\bar{n}_j+1)}\sigma_Z & \bigoplus_{j=1}^{n}(2\bar{n}_j+1)I_2 \end{bmatrix}. \tag{57}$$

We note that the symplectic matrix for the unitary evolution $\hat{S}_{A^n}\otimes I_{R^n}$ is given by

$$\begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix}. \tag{58}$$

6 Purity of a quantum state

The purity of a quantum state ρ is defined as $\mathrm{Tr}\{\rho^2\}$. We now show that $\mathrm{Tr}\{\rho^2\} \leq 1$. Consider the following spectral decomposition of the state ρ:

$$\rho = \sum_x \lambda_x |\phi_x\rangle\langle\phi_x|. \tag{59}$$

Then

$$\mathrm{Tr}\{\rho^2\} = \sum_x \lambda_x^2 \ . \tag{60}$$

Since $\lambda_x \leq 1 \Rightarrow \lambda_x^2 \leq 1$ and since $\sum_x \lambda_x = 1 \Rightarrow \sum_x \lambda_x^2 \leq 1$. Therefore, if a state is pure, then $\mathrm{Tr}\{\rho^2\} = 1$.

We now show that if $\mathrm{Tr}\{\rho^2\} = 1$, then the state is pure. Consider that

$$\begin{aligned} 1 &= \mathrm{Tr}\{\rho^2\} && (61)\\ &= \sum_x \lambda_x^2 \ . && (62) \end{aligned}$$

Moreover, $\mathrm{Tr}\{\rho\} = \sum_x \lambda_x = 1 \Rightarrow \mathrm{Tr}\{\rho\}^2 = \sum_{x,y} \lambda_x \lambda_y = 1$.

Consider the following chain of inequalities:

$$\begin{aligned} \Rightarrow 0 &= \mathrm{Tr}\{\rho^2\} - \mathrm{Tr}\{\rho\}^2 && (63)\\ &= \sum_x \lambda_x^2 - \left[\sum_{x,y} \lambda_x \lambda_y\right] && (64)\\ &= \sum_x \lambda_x^2 - \left[\sum_x \lambda_x^2 + \sum_{x\neq y} \lambda_x \lambda_y\right] && (65)\\ &= \sum_{x\neq y} \lambda_x \lambda_y \ . && (66) \end{aligned}$$

Since $\lambda_x, \lambda_y \geq 0$, the only possibility to satisfy (66) is that $\lambda_x = 1$ and $\lambda_y = 0, \forall y \neq x$. Thus, $\mathrm{Tr}\{\rho^2\} = 1$ implies that ρ is a pure state.

6.1 Purity of a Gaussian state

In this section, we calculate the purity for Gaussian states. From the Williamson decomposition of an n-mode Gaussian state as defined in (53) and from the fact that the purity is invariant under unitary transformations, we get

$$\mathrm{Tr}\{\rho^2\} = \prod_{j=1}^{n} \mathrm{Tr}\{\theta^2(\bar{n}_j)\} \ . \tag{67}$$

Consider the following chain of equalities:

$$\mathrm{Tr}\{\theta^2(\bar{n}_j)\} = \frac{1}{(\bar{n}_j+1)^2} \sum_{n=0}^{\infty} \left(\frac{\bar{n}_j}{\bar{n}_j+1} \right)^{2n} \tag{68}$$

$$= \frac{1}{(\bar{n}_j+1)^2} \frac{1}{1-\left(\bar{n}_j/(\bar{n}_j+1)\right)^2} \tag{69}$$

$$= \frac{1}{(\bar{n}_j+1)^2 - \bar{n}_j^2} \tag{70}$$

$$= \frac{1}{2\bar{n}_j+1} \tag{71}$$

$$= \frac{1}{\nu_j} \ , \tag{72}$$

where ν_j denotes the symplectic eigenvalue of $\theta(\bar{n}_j)$. The first equality follows from the definition of a thermal state as defined in (48). The second equality follows from the sum of an infinite geometric series.

Therefore,

$$\mathrm{Tr}\{\rho^2\} = \prod_{j=1}^{n} \frac{1}{\nu_j} \tag{73}$$

$$= \sqrt{\prod_{j=1}^{n} \frac{1}{\nu_j^2}} \tag{74}$$

$$= \frac{1}{\sqrt{\prod_{j=1}^{n} \nu_j^2}} \tag{75}$$

$$= \frac{1}{\mathrm{Det}(\sigma)} \ . \tag{76}$$

The last equality follows from (26) and from the fact that for any symplectic matrix S, $\mathrm{Det}(S) = 1$.

Therefore, the purity of a Gaussian state is

$$\mathrm{Tr}\{\rho^2\} = \frac{1}{\sqrt{\mathrm{Det}(\sigma)}} \ , \tag{77}$$

which implies that a Gaussian state is pure if and only if $\mathrm{Det}(\sigma) = 1$. Since $\nu_j \geq 1$, an equivalent condition for the purity of a Gaussian state is that all symplectic eigenvalues are equal to one.

7 Entropy of a Gaussian state

In this section, we find an expression for the von Neumann entropy of a Gaussian state.

The von Neumann entropy of a quantum state ρ is defined as

$$S(\rho) \equiv -\mathrm{Tr}\{\rho \ln \rho\} \ . \tag{78}$$

We begin by expressing a thermal state with the mean photon number $\bar{n}$ in the following form:

$$\theta(\bar{n}) = \frac{1}{\bar{n}+1} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{\bar{n}+1} \right)^n |n\rangle\langle n| \tag{79}$$

$$= \frac{1}{\bar{n}+1} \left(\frac{\bar{n}}{\bar{n}+1} \right)^{\hat{n}} . \tag{80}$$

Consider the following chain of equalities:

$$-\operatorname{Tr}\{\theta(\bar{n}) \ln \theta(\bar{n})\} = -\operatorname{Tr}\left\{ \theta(\bar{n}) \ln \frac{1}{\bar{n}+1} \left(\frac{\bar{n}}{\bar{n}+1} \right)^{\hat{n}} \right\} \tag{81}$$

$$= -\operatorname{Tr}\left\{ \theta(\bar{n}) \ln \left(\frac{1}{\bar{n}+1} \right) \right\} - \operatorname{Tr}\left\{ \theta(\bar{n}) \hat{n} \ln \left(\frac{\bar{n}}{\bar{n}+1} \right) \right\} \tag{82}$$

$$= \ln(\bar{n}+1) - \ln \left(\frac{\bar{n}}{\bar{n}+1} \right) \operatorname{Tr}\{\theta(\bar{n})\hat{n}\} \tag{83}$$

$$= \ln(\bar{n}+1) - \ln \left(\frac{\bar{n}}{\bar{n}+1} \right) \bar{n} \tag{84}$$

$$= (\bar{n}+1)\ln(\bar{n}+1) - \bar{n} \ln \bar{n} \tag{85}$$

$$\equiv g(\bar{n}) . \tag{86}$$

From unitary invariance and additivity of the von Neumann entropy, we get

$$S(\rho) = S\left(\bigotimes_{j=1}^{n} \theta(\bar{n}_j) \right) , \tag{87}$$

where ρ is an n-mode Gaussian state as defined in (53). Therefore,

$$S(\rho) = \sum_{j=1}^{n} S(\theta(\bar{n}_j)) \tag{88}$$

$$= \sum_{j=1}^{n} g(\bar{n}_j) \tag{89}$$

We now derive an alternative formula for the von Neumann entropy of faithful Gaussian states. Let

$$\rho = \frac{1}{\sqrt{\operatorname{Det}[(\sigma + i\Omega)/2)]}} \exp\left(-\frac{1}{2} (\hat{r} - \bar{r})^T H (\hat{r} - \bar{r}) \right) \tag{90}$$

$$= \hat{D}_{-\bar{r}} \left[\frac{\exp\left(-\frac{1}{2}\hat{r}^T H \hat{r}\right)}{\sqrt{\operatorname{Det}[(\sigma + i\Omega)/2)]}} \right] \hat{D}_{\bar{r}} \tag{91}$$

and let

$$\rho_0 = \frac{\exp\left(-\frac{1}{2}\hat{r}^T H \hat{r}\right)}{\sqrt{\operatorname{Det}[(\sigma + i\Omega)/2)]}} . \tag{92}$$

Then from unitary invariance of the von Neumann entropy, we get

$$
\begin{aligned}
S(\rho) &= S(\rho_0) & (93)\\
&= -\operatorname{Tr}\{\rho_0 \ln \rho_0\} & (94)\\
&= -\operatorname{Tr}\{\rho_0 \ln \frac{\exp(-\frac{1}{2}\hat{r}^T H \hat{r})}{\sqrt{\operatorname{Det}[(\sigma + i\Omega)/2]}}\} & (95)\\
&= -\operatorname{Tr}\{\rho_0 \ln \frac{1}{\sqrt{\operatorname{Det}[(\sigma + i\Omega)/2]}}\} - \operatorname{Tr}\{\rho_0 \ln \exp(-\frac{1}{2}\hat{r}^T H \hat{r})\} & (96)\\
&= \frac{1}{2} \ln \operatorname{Det}[(\sigma + i\Omega)/2] + \frac{1}{2} \operatorname{Tr}\{\rho_0 \hat{r}^T H \hat{r}\} \ . & (97)
\end{aligned}
$$

We now focus on the second term of the aforementioned equation.

$$
\begin{aligned}
\operatorname{Tr}\{\rho_0 \hat{r}^T H \hat{r}\} &= \operatorname{Tr}\{\rho_0 \sum_{j,k} \hat{r}_j H_{j,k} \hat{r}_k\} & (98)\\
&= \sum_{j,k} H_{j,k} \operatorname{Tr}\{\rho_0 \hat{r}_j \hat{r}_k\} & (99)\\
&= \frac{1}{2} \sum_{j,k} H_{j,k} \operatorname{Tr}\{\rho_0 (\{\hat{r}_j, \hat{r}_k\} + [\hat{r}_j, \hat{r}_k])\} & (100)\\
&= \frac{1}{2} \sum_{j,k} H_{j,k} (\sigma_{j,k} + i\Omega_{j,k}) & (101)\\
&= \frac{1}{2} \sum_{j,k} H_{j,k} \sigma_{j,k} - \frac{i}{2} \sum_{j,k} H_{j,k} \Omega_{k,j} & (102)\\
&= \frac{1}{2} \operatorname{Tr}\{H\sigma\} - \frac{i}{2} \operatorname{Tr}\{H\Omega\} & (103)\\
&= \frac{1}{2} \operatorname{Tr}\{H\sigma\}, & (104)
\end{aligned}
$$

where we used the fact that $\operatorname{Tr}\{H\Omega\} = 0$, which holds because H is symmetric and Ω is antisymmetric.

Therefore,

$$
S(\rho) = \frac{1}{2} \ln \operatorname{Det}[(\sigma + i\Omega)/2] + \frac{1}{4} \operatorname{Tr}\{H\sigma\} \ . \tag{105}
$$

Moreover, from (24) it follows that

$$
S(\rho) = \frac{1}{2} \ln \operatorname{Det}[(\sigma + i\Omega)/2] + \frac{1}{2} \operatorname{Tr}\{\operatorname{arccoth}(i\Omega\sigma) i\Omega\sigma\} \ . \tag{106}
$$

This latter expression is valid for pure Gaussian states, with the expression $\operatorname{Tr}\{\operatorname{arccoth}(i\Omega\sigma) i\Omega\sigma\}$ understood in a limiting sense.

References

[Ser17] Alessio Serafini. *Quantum Continuous Variables: A Primer of Theoretical Methods.* CRC Press, 2017.