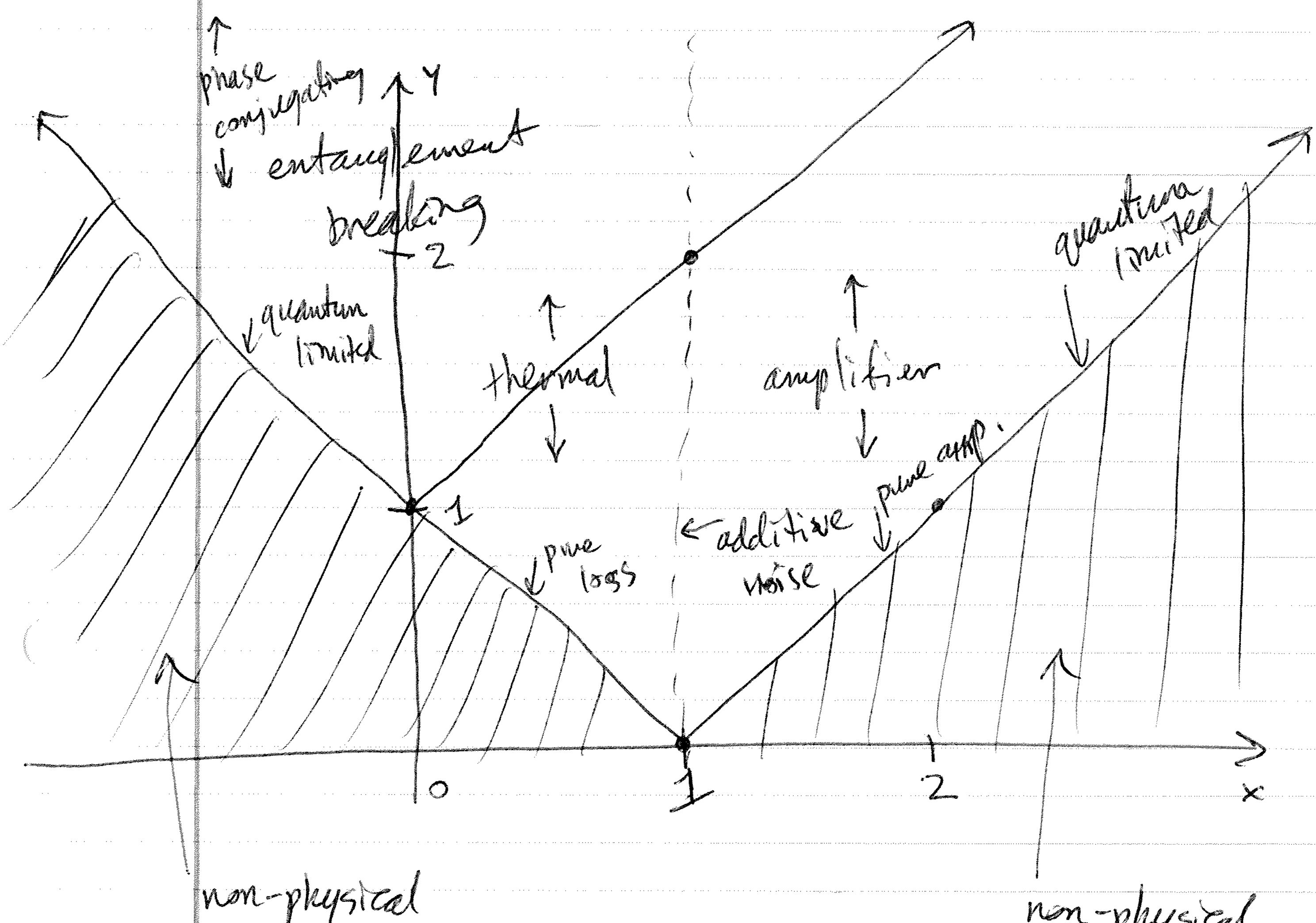
Lecture 22

Let us focus on the phase-insensitive ($\overset{\text{phase}}{\text{covariant}}$ & contravariant) channels in the Holevo classification, as they have many interesting properties & relations

Class	Scale Matrix	Noise Matrix
A_1	$X = 0 = \sqrt{x}\, I$ w/ $x=0$	$Y = yI \quad y \geq 1$
B_2	$X = I = \sqrt{x}\, I$ w/ $x=1$	$Y = yI \quad y \geq 0$
C thermal & amplifier	$X = \sqrt{x}\, I$, $x \in (0,1) \cup (1,\infty)$	$Y = yI$, $y \geq \lvert x-1 \rvert$
D	$X = \sqrt{\lvert x \rvert}\, \sigma_Z$, $x < 0$ (i.e., $\text{Det}(X) = x$)	$Y = yI$, $y \geq 1 + \lvert x \rvert = \lvert x - 1 \rvert$



region of physical maps is

$$y \geq |x-1|$$

(i.e., $iX\Omega X^T + Y \geq i\Omega$)

Channels are entanglement-breaking (i.e., no ability to generate entanglement)

when $y \geq |x|+1$

"too much noise"

Recall that Choi state of single-mode channel has covariance matrix

$$\begin{bmatrix} \cosh(2r)\, I_2 & \sinh(2r)\, \sigma_z X^T \\ \sinh(2r)\, X \sigma_z & \cosh(2r)\, X X^T + Y \end{bmatrix}$$

so for phase-ins., we get

$$\begin{bmatrix} \cosh(2r)\, I_2 & \sqrt{|x|} \sinh(2r)\, \sigma_z^{H(x)} \\ \sqrt{|x|} \sinh(2r)\, \sigma_z^{H(x)} & \left[\cosh(2r)\,(|x|) + y\right] I_2 \end{bmatrix}$$

where $H(x)$ is Heaviside step function

$$H(x) = \begin{cases} 1 : x \geq 0 \\ 0 : x < 0 \end{cases}$$

can show that any two-mode state in standard form

$$\begin{bmatrix} a I_2 & c\, \sigma_z^d \\ c\, \sigma_z^d & b I_2 \end{bmatrix}$$

for $d \in \{0,1\}$ is separable if & only if

$$c \leq \sqrt{(a-1)(b-1)}$$

For our case, this reduces to testing this inequality w/ $r > 0 \Rightarrow \cosh(2r) > 1$

$$a = \cosh(2r)$$
$$b = \cosh(2r)|x| + y$$
$$c = \sqrt{|x|}\sinh(2r)$$

condition is then

$$\sqrt{|x|}\sinh(2r) \le \sqrt{(\cosh(2r)-1)(\cosh(2r)|x| + y - 1)}$$
$$= \sqrt{(\cosh^2(2r)|x| - \cosh(2r)|x| + \cosh(2r)y - y - \cosh(2r) + 1)}$$
$$= \sqrt{(\cosh^2(2r)-1)|x| + |x| - \cosh(2r)|x| + \cosh(2r)y - y - \cosh(2r) + 1)}$$
$$= \sqrt{\sinh^2(2r)|x| + [1-\cosh(2r)][|x| - y + 1]}$$

square both sides & get reduction to

$$0 \le [1 - \cosh(2r)][|x| - y + 1]$$

since $\cosh(2r) > 1$ for $r > 0$

reduces to $y \ge |x| + 1$ as claimed

Channel decompositions

If a channel is phase covariant ($x \geq 0$), it can be realized by serial concatenation of pure loss followed by pure amplifier!

$$\mathcal{N}_{x,y} = \mathcal{A}_{G,0} \circ \mathcal{L}_{\eta,0}$$

where $x = \eta G$

$$y = G(1-\eta) + G - 1$$

or inverse equations are

$$G = \frac{x+1+y}{2}$$

$$\eta = \frac{x}{G} = \frac{2x}{x+1+y}$$

Check whether

$\eta \in (0,1)$ & $G > 1$

Use $y \geq |x-1|$

$$G = \frac{x+1+y}{2} \geq \frac{x+1+|x-1|}{2}$$

if $x \geq 1$, then

$$= \frac{x+1+x-1}{2} \geq \frac{2x}{2} \geq 1$$

if $x \leq 1$, then

$$= \frac{x+1+1-x}{2} = 1$$

$$\eta = \frac{2x}{x+1+y} \geq 0 \quad \text{b/c} \quad x, y \geq 0$$

$(y \geq |x-1|)$

Also, $\dfrac{2x}{x+1+y} \leq \dfrac{2x}{x+1+|x-1|}$

if $x \geq 1$, then $= \dfrac{2x}{x+1+x-1} = 1$

if $x \leq 1$, then $\dfrac{2x}{x+1+1-x} = x \leq 1$

done

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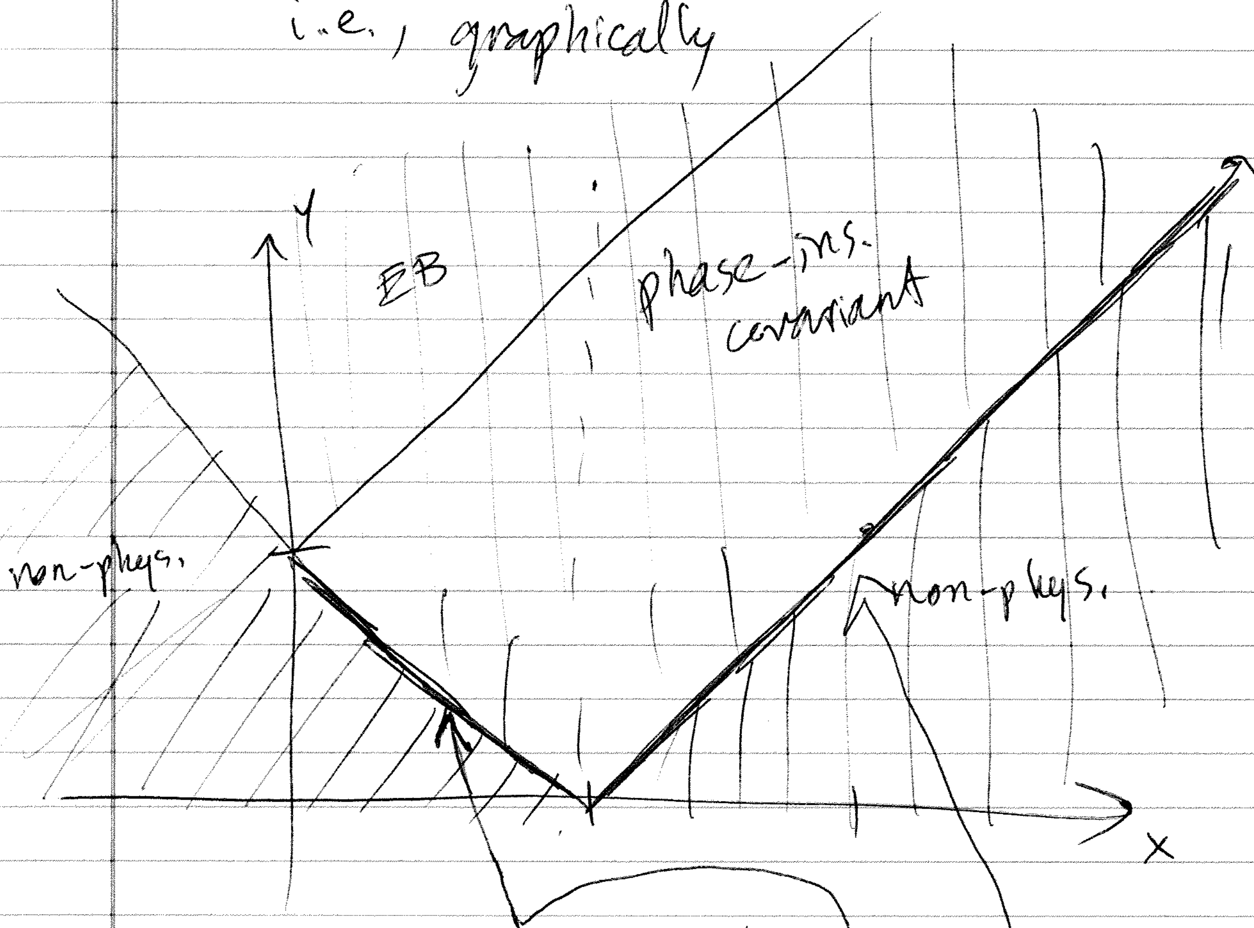
i.e., graphically



1) combine ~~#~~ pureloss

2) w/ pure amp.

3) to realize any channel w/ $x \geq 0$

for phase-covariant channels (7)

Opposite decomposition if channel is not entanglement-breaking

i.e., not only $y \geq |x-1|$

but also $y < |x|+1 = x+1$ (not EB)

$$\mathcal{N}_{x,y} = \mathcal{L}_{\eta,0} \circ \mathcal{A}_{G,0}$$

where $x = \eta G$ &

$y = \eta(G-1) + 1 - \eta$

or inverse equations are

$$\eta = \frac{x+1-y}{2}, \quad G = \frac{x}{\eta} = \frac{2x}{x+1-y}$$

Check whether $\eta \in (0,1)$ & $G > 1$

$\eta > 0$ b/c $\eta = \frac{x+1-y}{2}$ & $y < x+1$ (not EB)

$\eta \leq 1$ b/c

$$\eta = \frac{x+1-y}{2} \leq 1 \iff x+1-y \leq 2$$
$$\iff x-y \leq 1$$

$\Leftrightarrow$ $x-1 \leq y$ & we have

$$y \geq |x-1|$$

from CPTP cond.

$G > 1$ b/c

$$G = \frac{2x}{x+1-y} \geq \frac{2x}{x+1-|x-1|}$$

if $x \geq 1$ then $= \dfrac{2x}{x+1-(x-1)}$

$= \dfrac{2x}{2} = x \geq 1$

if $x < 1$ then $= \dfrac{2x}{x+1-(1-x)}$

$= \dfrac{2x}{2x} = 1$

done.

Any additive-noise channel w/ $x=1$

& $y \geq 0$

can be realized by pure loss η

followed by pure amplifier w/ gain $\frac{1}{\eta}$

$\mathcal{T}_y = \mathcal{A}_{1/\eta, 0} \circ \mathcal{L}_{\eta, 0}$

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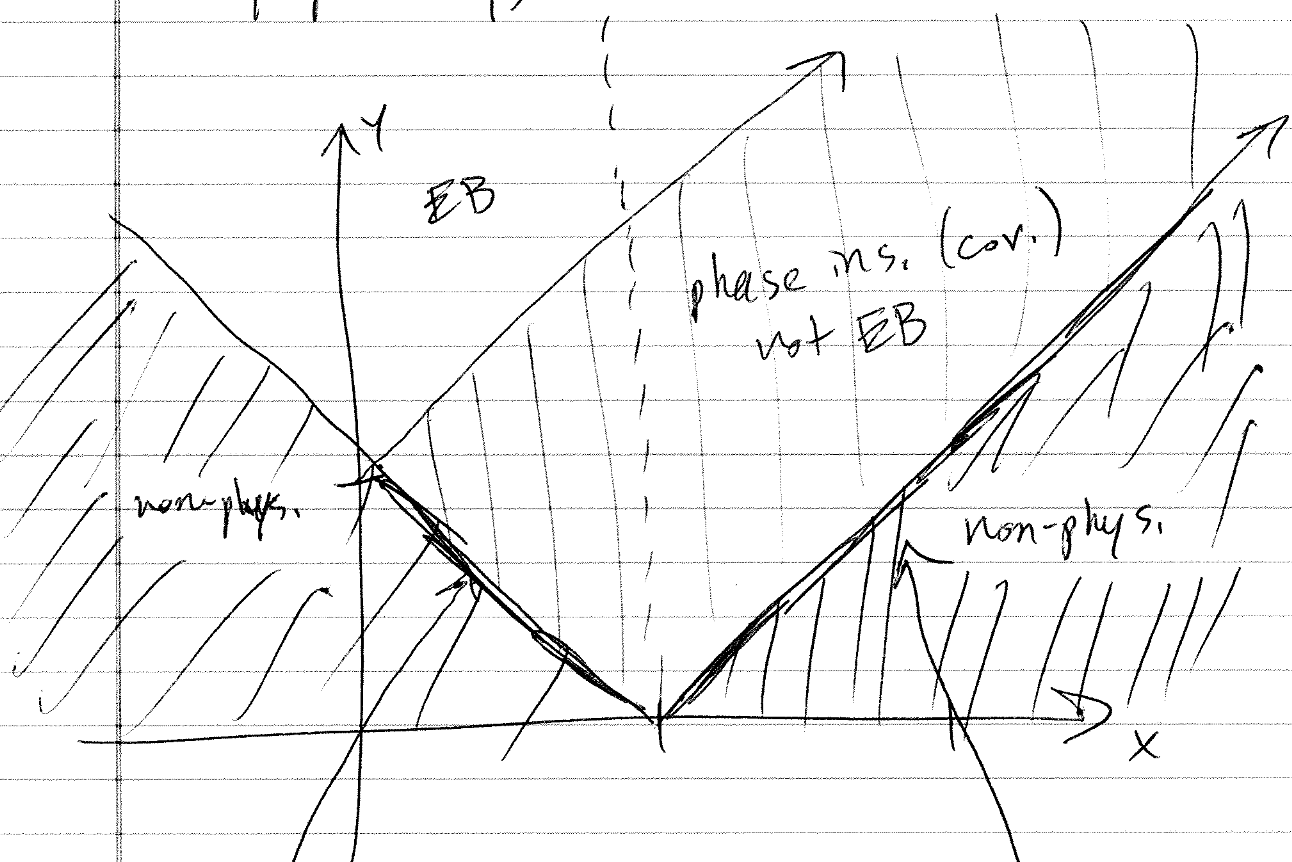
i.e.,
graphically,



1) combine pure amp

2) w/ pure loss

3) to get any phase-covariant channel
that is not EB.

$$\sigma \xrightarrow{\mathcal{L}_{\eta,0}} \eta\sigma + 1-\eta$$

$$\xrightarrow{\mathcal{A}_{1/\eta,0}} \frac{1}{\eta}(\eta\sigma + 1-\eta) + \frac{1}{\eta} - 1$$

$$= \sigma + \frac{1-\eta}{\eta} + \frac{1}{\eta} - 1$$

$$= \sigma + 2\left(\frac{1}{\eta} - 1\right)$$

$$\text{so } y = 2\left(\frac{1}{\eta} - 1\right)$$

or inverse:

$$\eta = \frac{1}{\frac{y}{2}+1} \in (0,1)$$

for $y > 0$

How does the partial transposition operation come into play in Gaussian QI?

Consider taking transposition in photon number basis, i.e.,

$$T(\cdot) = \sum_{n,m} |n\rangle\langle m| (\cdot) |n\rangle\langle m|$$

for creation & annihilation op's,)

$\hat{a}^T = \hat{a}^\dagger$ because

$$\hat{a} = \sum_{n=0}^{\infty} \sqrt{n+1}\, |n\rangle\langle n+1|$$

so then

$$\hat{x}^T = \left(\frac{\hat{a}+\hat{a}^\dagger}{\sqrt{2}}\right)^T = \frac{\hat{a}^T + \hat{a}^{\dagger T}}{\sqrt{2}}$$

$$= \frac{\hat{a}^\dagger + \hat{a}}{\sqrt{2}} = \hat{x}$$

but

$$\hat{p}^T = \left(\frac{\hat{a}-\hat{a}^\dagger}{\sqrt{2}i}\right)^T = \frac{\hat{a}^T - (\hat{a}^\dagger)^T}{\sqrt{2}i}$$

$$= \frac{\hat{a}^\dagger - \hat{a}}{\sqrt{2}i} = -\hat{p}$$

for covariance matrix,
this means that on a single mode state

$$\sigma_{ij} = \mathrm{Tr}\left[\{\hat{r}_i, \hat{r}_j\}\, T(\rho)\right]$$

$$= \mathrm{Tr}\left[T(\hat{r}_i \hat{r}_j + \hat{r}_j \hat{r}_i)\, \rho\right]$$

$\Rightarrow$ w/ $\hat{r}_i \in \{\hat{x}, \hat{p}\}$

& $T(\hat{x}) = \hat{x} \quad T(\hat{p}) = -\hat{p}$

$$\mathrm{Tr}\left[\, T(\hat{r}_i \hat{r}_j + \hat{r}_j \hat{r}_i)\ \rho\right]$$
$$= \mathrm{Tr}\left[(\hat{r}_i^T \hat{r}_j^T + \hat{r}_j^T \hat{r}_i^T)\ \rho\right]$$

$$\Rightarrow \quad \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \rightarrow \begin{bmatrix} \sigma_{11} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{22} \end{bmatrix}$$

or $\quad \sigma \rightarrow \sigma_z \, \sigma \, \sigma_z$

So when taking partial transpose, operation is then

$$\sigma_{AB} \rightarrow \begin{bmatrix} I & \\ & \sigma_z \end{bmatrix} \sigma_{AB} \begin{bmatrix} I & \\ & \sigma_z \end{bmatrix}$$

note that σ_z is

<u>antisymplectic</u> operation

(i.e., $\sigma_z \Omega \sigma_z^T = -\Omega$)

when taking transpose operation @ output of single-mode channel, the effect is

$$X \sigma X^T + Y \rightarrow \sigma_z X \sigma X^T \sigma_z + \sigma_z Y \sigma_z$$

this need not be physical!

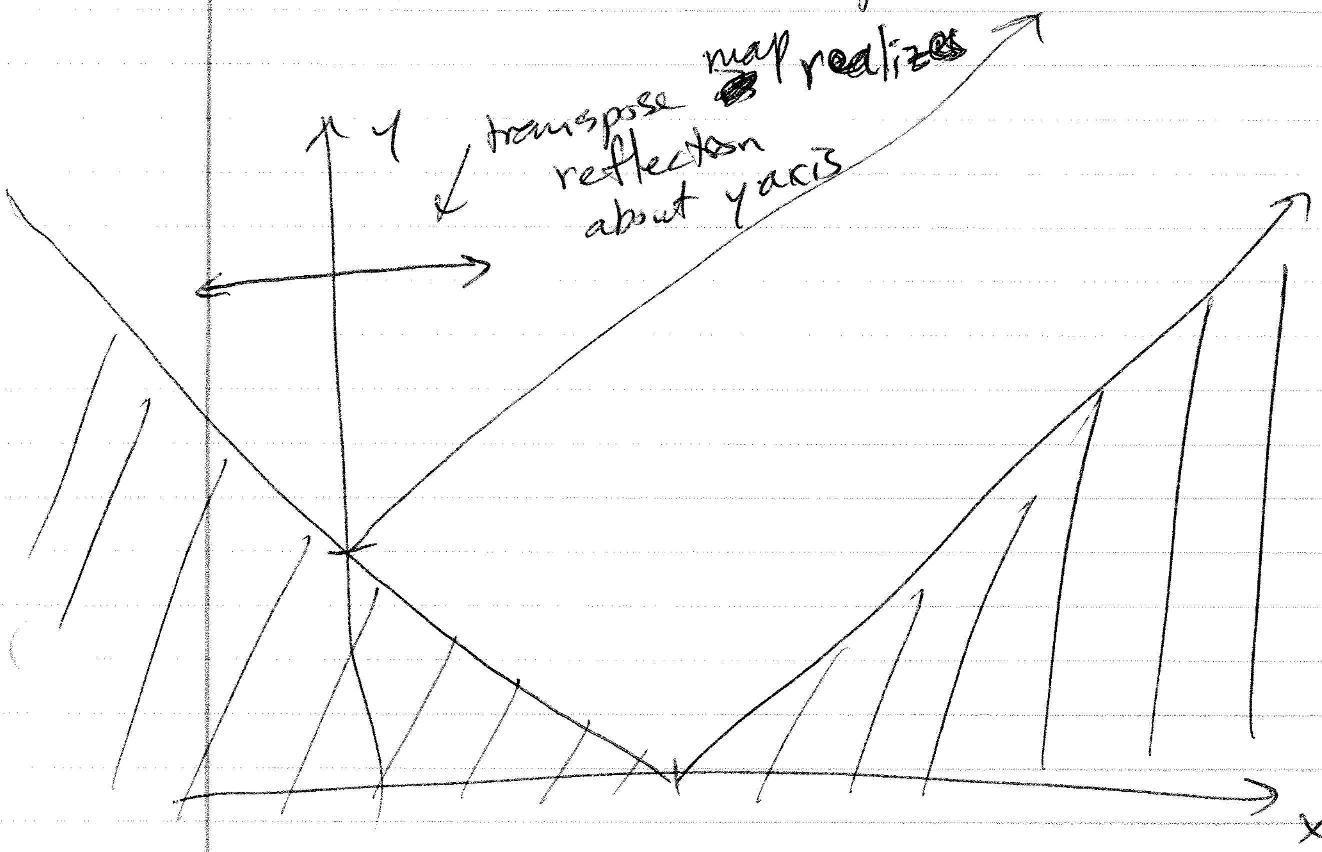
However, consider what happens when taking transpose of phase conjugating channel

$$X = \sqrt{|x|}\,\sigma_z \qquad Y = y I_2$$

$\Rightarrow$ new channel has

$$X' = \sqrt{|x|}\, I_2 \quad \& \quad Y' = y I_2$$

i.e., becomes thermal or amplifier channel in terms of picture



notice how transpose ~~gives~~ leads to physical channel when acting on those that are EB i.e.

$$y \geq |x|+1$$

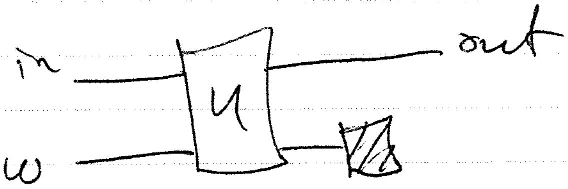
amounts to $x \to -x$ + EB condition unchanged.

this realizes symmetry of EB region about y axis

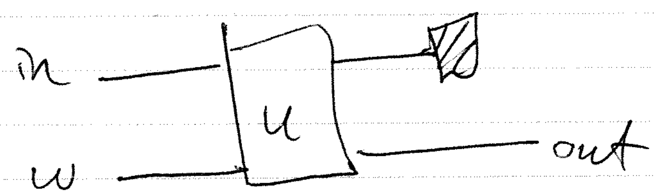
Other transformations:

Weak complementary

Given channel $\mathcal{N}$ realized by



weak complementary is



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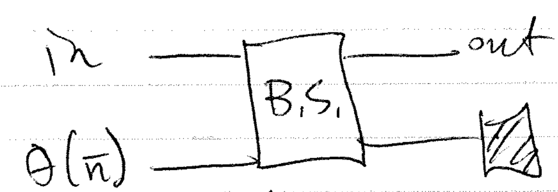
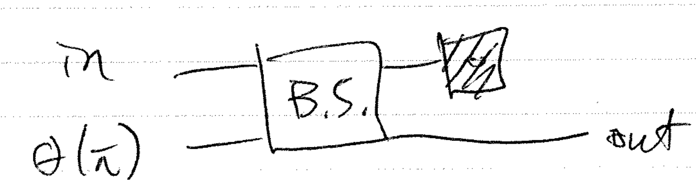
thermal channel



weak complementary is then



if original thermal channel has $x \in (0,1)$ & $y \geq 0$, ~~then weak complement has $x' = 1-x$~~

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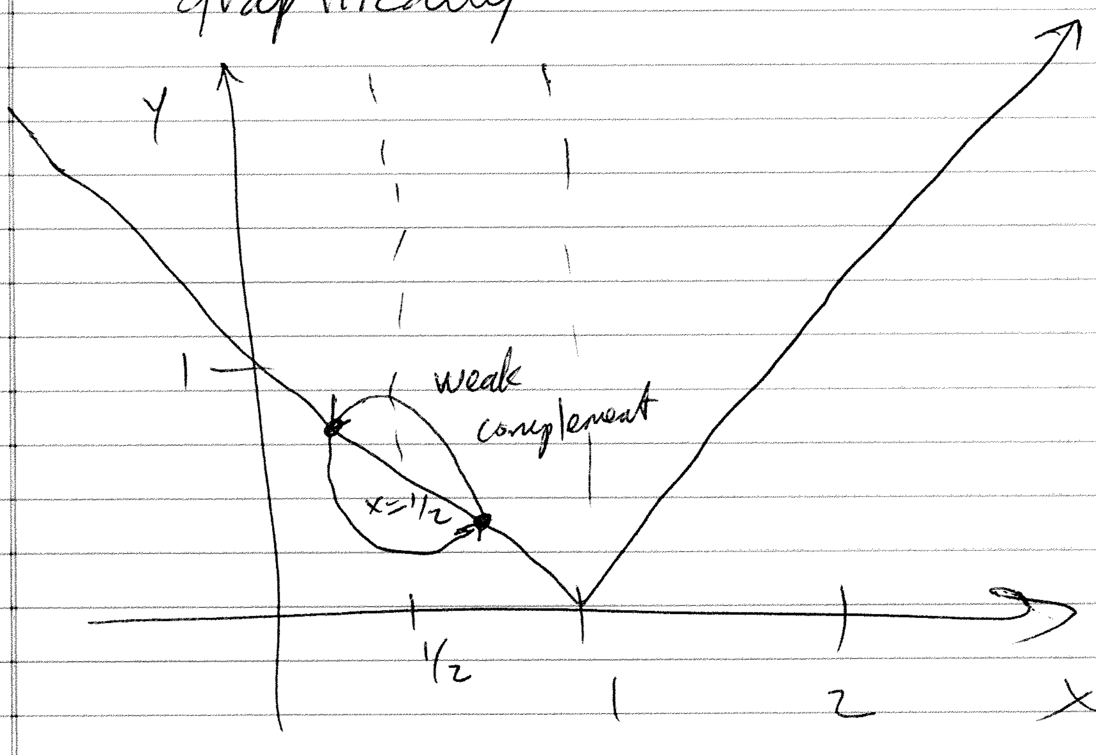
then weak complement has

$$x' = 1 - x \quad \&$$

$$y' = \frac{x}{1-x}\, y$$

~~for amplifier~~

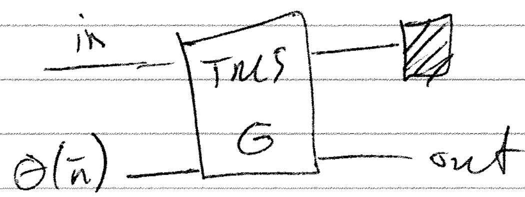
graphically



(16)

for amplifier channel

weak complement is then phase conjugator



if original channel has

$$x > 1 \quad \& \quad y \geq 0$$

then weak complement has

$$x' = 1 - x \quad \& \quad y' = y\left(\frac{x+1}{|x-1|}\right)$$

graphically,

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