Lecture 16

Examples

Displacement operators

$$\hat{D}_r = e^{i r^T \Omega \hat{r}} \qquad \text{for } r \in \mathbb{R}^{2n}$$

Effect on mean vector $\bar{r}$ & cov. matrix σ of a generic state ρ is

$$\begin{aligned}
\mathrm{Tr}[\hat{r}\, \hat{D}_r^\dagger \rho \hat{D}_r] &= \mathrm{Tr}\{\hat{D}_r \hat{r} \hat{D}_r^\dagger \rho\} \\
&= \mathrm{Tr}[(\hat{r} + r)\rho] \\
&= \mathrm{Tr}[\hat{r}\rho] + r \\
&= \bar{r} + r
\end{aligned}$$

Cov. matrix does not change.

For Gaussian states, evolution completely characterized by $\bar{r} \to \bar{r} + r$, $\sigma \to \sigma$

- Since 1st moments can be changed arbitrarily, no properties depending only on spectrum of a generic state can depend on 1st moments.

- Examples include entropies & Renyi entropies.

- Furthermore, multimode displacements are tensor products of local displacements.

- So any quantity invariant under local unitaries does not depend on 1st moments.

- Examples include mutual information & entanglement measures

- linear displacements can be implemented by passive quadratic operations acting on the input modes & external modes prepared in strong coherent states.

<u>Symplectic operations</u>

effect of purely quadratic Hamiltonian is unitary $e^{\frac{i}{2}\hat{r}^T H \hat{r}}$ & effect on mean vector & cov. matrix of generic state is

$$\bar{r} \to S\bar{r} \qquad \text{where } S = e^{\Omega H}$$

$$\sigma \to S\sigma S^T$$

then $\bar{r} \to S\bar{r}$

$\sigma \to S\sigma S^T$

completely characterizes evolution of a Gaussian state.

<u>Singular value decomposition of a symplectic transformation</u>

Any $2n \times 2n$ symplectic matrix S can be decomposed as

$$S = O_1 Z O_2$$

where O_1 & O_2 are symplectic & orthogonal [illegible] &

$$Z = \bigoplus_{j=1}^{n} \begin{bmatrix} z_j & 0 \\ 0 & z_j^{-1} \end{bmatrix}$$

for $z_j \geq 1$

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Proof: use xp-ordering $[\hat{x}_1 \cdots \hat{x}_n \; \hat{p}_1 \cdots \hat{p}_n]$

w/ $\Omega \to J = \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix}$

Start w/ polar decomposition of S.

First, we know that any symplectic matrix S is invertible.

It has a polar decomposition as

$$S = P\,O$$

w/ $P = (SS^T)^{1/2}$ ← positive definite

$O = (SS^T)^{-1/2} S$ ← orthogonal

O is orthogonal because

$$OO^T = (SS^T)^{-1/2}(SS^T)(SS^T)^{-1/2} = I$$

It is also <u>unique</u>

To see this, Suppose that

$$S = P_1 O_1 = P_2 O_2$$

$\Rightarrow P_1 = P_2 O_2 O_1^T$

$\Rightarrow P_1^2 = P_2 O_2 O_1^T P_1$

Similarly, $P_2 = P_1 O_1 O_2^T$

$\Rightarrow P_2^2 = P_1 O_1 O_2^T P_2$

P_2 symmetric $\Rightarrow$ transpose to get

$$P_2^2 = P_2 O_2 O_1^T P_1$$

since square root of positive definite matrix is unique, then

$P_1 = P_2$

$\Rightarrow O_1 = O_2$ also

Consider that

$$S^T J S = J \Rightarrow \quad S = J^{-1} S^{-T} J$$

Since $\quad S = PO$

$$\Rightarrow \quad S = J^{-1} (PO)^{-T} J$$

$$= J^{-1} P^{-T} O^{-T} J$$

$$= \underbrace{J^{-1} P^{-T} J}_{\text{this is positive definite}} \underbrace{J^{-1} O^{-T} J}_{\text{this is orthogonal}}$$

uniqueness of polar decomposition

$\Rightarrow$

$$P = J^{-1} P^{-T} J \qquad O = J^{-1} O^{-T} J$$

$$\Rightarrow P J P^T = J \qquad O J O^T = J$$

So P is symplectic, as well as O.

Now we can use this to arrive @ conclusion for symplectic SVD.

P is positive definite & thus can be diagonalized w/ strictly positive eigenvalues.

Suppose that v is an eigenvector of P w/ eigenvalue λ:

$$Pv = \lambda v$$

Then Jv is an eigenvector of P w/ eigenvalue λ^{-1}:

$$P\underline{Jv} = \lambda^{-1}\underline{Jv}$$

because

$$P J v = \underbrace{P J P^T}_{} P^{-T} v$$

$$= J P^{-1} v \qquad \text{(}P\text{ symplectic \& symmetric)}$$

$$= \lambda^{-1} J v$$

So this means that eigenvalues of P come in pairs. Also, recall that $v^T J v = v^T J v = 0 \quad \forall v$

so $v \perp J v$.

Let V be $2n \times n$ matrix of 1st n eigenvectors of P

then eigendecomposition of P is

$$P = \underbrace{\begin{bmatrix} JV & V \end{bmatrix}}_{} \begin{bmatrix} \lambda_1^{-1} & & & & & \\ & \ddots & & & & \\ & & \lambda_n^{-1} & & & \\ & & & \lambda_1 & & \\ & & & & \ddots & \\ & & & & & \lambda_n \end{bmatrix} \begin{bmatrix} V^T J^T \\ V^T \end{bmatrix}$$

call this K. this matrix is symplectic

$\hookrightarrow$ $K^T J K = J$ because

$$K^T J K$$

$$= \begin{bmatrix} V^T J^T \\ V^T \end{bmatrix} J \begin{bmatrix} JV & V \end{bmatrix}$$

$$= \begin{bmatrix} V^T J^T J = V^T \\ V^T J \end{bmatrix} \begin{bmatrix} JV & V \end{bmatrix}$$

$$= \begin{bmatrix} V^T J V & V^T V \\ V^T J J V & V^T J V \end{bmatrix}$$

$$= \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} = J$$

orthogonality of $v^T J v$ $\forall v$

$V^T V = I$ due to orthonormality of V

Final conclusion

$$S = P\,O \qquad \text{w/ } P \text{ pos. def. \& symplectic}$$

$$= K \begin{bmatrix} \Lambda^{-1} & \\ & \Lambda \end{bmatrix} K^T O \qquad \text{\& } O \text{ orthogonal \& symplectic}$$

w/ K orthogonal & symplectic

& $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$

w/ $\lambda_i \geq 1$

Then in statement of theorem,

take $O_1 = K$, $Z = \begin{bmatrix} \Lambda^{-1} & \\ & \Lambda \end{bmatrix}$

$O_2 = K^T O$