PHYS 7895 Spring 2019
Gaussian Quantum Information
Homework 3

Due Friday, 26 April 2019, by 4pm in Nicholson 447

1. Prove that the symplectic evolution realized by the Hamiltonian matrix $H_1 \oplus H_2$ (with H_1 and H_2 real symmetric matrices) is given by $e^{\Omega H_1} \oplus e^{\Omega H_2}$.

2. Prove that the mean vector of the two-mode squeezed vacuum state

$$\frac{1}{\cosh(r)} \sum_{n=0}^{\infty} \tanh(r)^n |n\rangle|n\rangle \tag{1}$$

for $r > 0$, is equal to zero. Prove that its covariance matrix is given by

$$\begin{bmatrix} \cosh(r) I_2 & \sinh(r)\sigma_Z \\ \sinh(r)\sigma_Z & \cosh(r) I_2 \end{bmatrix}. \tag{2}$$

3. Prove that the single-mode thermal state is the only Gaussian state that is phase invariant (i.e., $\rho = e^{-i\hat{n}\phi}\rho e^{i\hat{n}\phi}$ for all $\phi \in \mathbb{R}$).

4. The entropy variance of a quantum state ρ is given by the formula:

$$\mathrm{Tr}[\rho(-\log\rho - S(\rho))^2], \tag{3}$$

where $S(\rho)$ denotes the quantum entropy. Find a formula for the entropy variance of an n-mode Gaussian state in terms of its symplectic eigenvalues.

5. Prove that m modes are necessary and sufficient for a Gaussian purification of a Gaussian state having m symplectic eigenvalues strictly greater than one.

6. BONUS: Find a formula for quantum χ^2 divergence of Gaussian states ρ and σ in terms of their mean vectors and covariance matrices. Recall that the quantum χ^2 divergence is defined for $\alpha \in [0, 1]$ as

$$\chi^2_\alpha(\rho, \sigma) = \mathrm{Tr}[(\rho - \sigma)\sigma^{-\alpha}(\rho - \sigma)\sigma^{\alpha-1}] \tag{4}$$

$$= \mathrm{Tr}[\rho\sigma^{-\alpha}\rho\sigma^{\alpha-1}] - 1. \tag{5}$$

7. BONUS: Find a formula for the fidelity of Gaussian states ρ and σ in terms of their mean vectors and covariance matrices by using the method of characteristic functions and ensuing overlap formulas.