Lecture 19

Recall Bessel's equation

$$x^2y'' + xy' + (x^2-p^2)y = 0$$

where $p \in \mathbb{R}$ is a constant

We'll start w/ $p \in \{0, 1, 2, \ldots\}$ & then generalize.

- Note that $(xy')' = y' + xy''$ so that Bessel's equation becomes

$$x[xy']' + (x^2-p^2)y = 0$$

First suppose that $p=0$.

Then

$$x[xy']' + x^2y = 0$$

$$\Rightarrow [xy']' + xy = 0$$

Suppose solution has the Frobenius form:

$$y = \sum_{n=0}^{\infty} a_n x^{n+s}$$

$$y' = \sum_{n=0}^{\infty} a_n (n+s) x^{n+s-1}$$

$$xy' = \sum_{n=0}^{\infty} a_n (n+s) x^{n+s}$$

$$[xy']' = \sum_{n=0}^{\infty} a_n (n+s)^2 x^{n+s-1}$$

$$xy = \sum_{n=0}^{\infty} a_n x^{n+s+1}$$

plugging in, we get

$$\sum_{n=0}^{\infty} a_n (n+s)^2 x^{n+s-1} + \sum_{n=0}^{\infty} a_n x^{n+s+1} = 0$$

Now relabel the second sum

$n+1=m-1$

$$\Rightarrow \quad m=n+2, \quad n=m-2$$

$n\in\{0,1,\dots\} \Rightarrow m\in\{2,3,\dots\}$

$$\Rightarrow \sum_{n=0}^{\infty} a_n (n+s)^2 x^{n+s-1} + \sum_{n=2}^{\infty} a_{n-2} x^{n+s-1} = 0$$

$n=0, 1$ special

$n=0 \Rightarrow a_0 s^2 = 0 \Rightarrow s=0$

$n=1 \Rightarrow a_1 (1+s)^2 = 0 \Rightarrow s=-1$

these correspond to 2 independent solutions.

– Set $s=0$ first,

this implies that $a_1=0$

For $n\geq 2$, we have the recursion

SS0

$$a_n n^2 + a_{n-2} = 0$$

$$\Rightarrow a_n = -\frac{1}{n^2} a_{n-2}$$

$$\Rightarrow a_{n+2} = \frac{-1}{(n+2)^2} a_n \quad \text{for } n \geq 0$$

all odd terms vanish b/c $a_1 = 0$

$$\Rightarrow a_2 = \frac{-1}{2^2} a_0$$

$$a_4 = \frac{-1}{4^2} \cdot a_2 = \frac{1}{4^2 2^2} a_0$$

$$\vdots$$

$$a_{2m} = \frac{(-1)^m}{(2m)^2 (2m-2)^2 \cdots 2^2} a_0$$

$$= \frac{(-1)^m}{2^{2m} \{m(m-1) \cdots 1\}^2} a_0$$

$$= \frac{(-1)^m}{2^{2m} [m!]^2} a_0$$

(5)

solution is then, for $s=0$ & $p=0$

$$y(x) = a_0 \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m}[m!]^2} x^{2m}$$

$$= a_0 \sum_{m=0}^{\infty} \frac{(-1)^m}{[m!]^2} \left(\frac{x}{2}\right)^{2m}$$

- Don't recognize the series.
- Also, it does not truncate & so it is not merely a polynomial
- Taking the boundary condition

$$y(0) = 1 \Rightarrow a_0 = 1, \text{ so}$$

then

$$y(x) = J_0(x) \equiv \sum_{n=0}^{\infty} \frac{(-1)^n}{[n!]^2} \left(\frac{x}{2}\right)^{2n}$$

$\boxed{\text{Show in Mathematica versus cosine}}$

Now consider when $p \neq 0$ but positive integer

$$x(xy')' + (x^2 - p^2)y = 0$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+s}$$

y' by same steps

$$(xy')' = \sum_{n=0}^{\infty} a_n (n+s)^2 x^{n+s-1}$$

$$\Rightarrow \quad x(xy')' = \sum_{n=0}^{\infty} a_n (n+s)\, x^{n+s}$$

$$x^2 y = \sum_{n=0}^{\infty} a_n x^{n+s+2}$$

$$-p^2 y = \sum_{n=0}^{\infty} (-p^2) a_n x^{m+s}$$

plug in to find that 7

$$\Rightarrow \sum_{n=0}^{\infty} a_n (n+s)^2 x^{n+s} \quad (I)$$

$$+ \sum_{n=0}^{\infty} a_n x^{n+2+s} \quad (II)$$

$$+ \sum_{n=0}^{\infty} a_n (-p^2) x^{n+s} \quad (III)$$

re-index 2nd one

$$(II) = \sum_{n=0}^{\infty} a_n x^{\overbrace{n+2}^{m}+s}$$

$$= \sum_{m=2}^{\infty} a_{m-2} x^{m+s}$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n (n+s)^2 x^{n+s}$$

$$+ \sum_{n=2}^{\infty} a_{n-2} x^{n+s}$$

$$+ \sum_{n=0}^{\infty} (-p^2) a_n x^{n+s}$$

treat $n=0$ & 1 differently

$n=0 \Rightarrow \quad (0+s)^2 a_0 +$

$$(-p^2)\, a_0 = 0$$

$$\Rightarrow s^2 - p^2 = 0 \Rightarrow s = \pm p$$

gives two <u>independent</u> solutions

consider $s = +p$ &

$n=1$

$$\Rightarrow a_1 (1+s)^2 - p^2 a_1 = 0$$

$$\Rightarrow \left[(1+p)^2 - p^2\right] a_1 = 0$$

$$\Rightarrow \left[1 + 2p + p^2 - p^2\right] a_1 = 0$$

$$\Rightarrow \left[1 + 2p\right] a_1 = 0$$

$$\Rightarrow a_1 = 0$$

Now $s = +p$ & $n \geq 2$

$$\Rightarrow (n+s)^2 a_n + a_{n-2} - p^2 a_n = 0$$

$$\Rightarrow \left[(n+p)^2 - p^2\right] a_n = -a_{n-2}$$

$$\Rightarrow \left[n^2 + 2np + p^2 - p^2\right] a_n = -a_{n-2}$$

$$\Rightarrow \left[n^2 + 2pn\right] a_n = -a_{n-2}$$

$$\Rightarrow a_n = \frac{-a_{n-2}}{n(n+2p)}$$

$$\Rightarrow a_{n+2} = \frac{-a_n}{(n+2)(n+2+2p)}$$

w/ $a_1 = 0 \Rightarrow$ all odd $a_{2m+1} = 0$

Then $a_{2m+2} = \dfrac{-1}{(2m+2)(2m+2+2p)} a_{2m}$

$$\Rightarrow \quad a_{2m+2} = \frac{-1}{4(m+1)(m+1+p)} a_{2m}$$

Recursion relation:

$m=0$

$$a_2 = \frac{-1}{4(1)(1+p)} a_0$$

$m=1$

$$a_4 = \frac{-1}{4(2)(2+p)} a_2 = \frac{a_0}{4^2 (2\cdot 1)(2+p)(1+p)}$$

$m=2$

$$a_6 = \frac{-1}{4^3 (3\cdot 2\cdot 1)(3+p)(2+p)(1+p)} a_0$$

Note that $\dfrac{(3+p)!}{p!} = (3+p)(2+p)\cdot(1+p)$

then

$$a_{2m} = \frac{(-1)^m \, p!}{4^m \, m! \, (m+p)!} a_0$$

Put into the series to get

$$y(x) = a_0 \sum_{m=0}^{\infty} \frac{(-1)^m p!}{2^{2m}\, m!\, m+p!}\, x^{2\underset{n}{m}+\underset{s}{p}}$$

p is a constant, so then multiply by $\frac{2^p}{2^p}$ to get

$$y(x) = a_0 2^p p! \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+p)!} \left(\frac{x}{2}\right)^{2n+p}$$

can choose boundary condition

$$a_0 = a_0(p) = \frac{1}{2^p p!}$$

as the final boundary condition to get

$$y_p(x) = J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, n+p!} \left(\frac{x}{2}\right)^{2n+p}$$

This recovers the previous result $p=0$ & holds for $p = 0, 1, 2, \ldots$

- Bessel function of the 1st kind of integer order p.

- Recall if $p \neq 0, 1, 2, 3, \ldots$ we can define $n+p! = \Gamma(n+p+1)$ & the whole proof goes through.

- then solution is

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+p+1)} \left(\frac{x}{2}\right)^{2n+p}$$

for $p \geq 0$.

Recall that there is a second independent solution w/ $s = -p$ instead of $s = p$.

However we can just substitute $p \to -p$ to get

$$J_{-p}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n-p+1)} \left(\frac{x}{2}\right)^{2n-p}$$

for $p > 0$

If $-p = -1, -2, -3$

then $\Gamma(0-2+1) = \Gamma(-1) = \infty$

$$\Rightarrow \frac{1}{\Gamma(-1)} = 0$$

The general solution to Bessel's equation is as follows:

$$y_p(x) = A\,J_p(x) + B\,J_{-p}(x)$$

We can write

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+2)} \left(\frac{x}{2}\right)^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+1)!} \left(\frac{x}{2}\right)^{2n+1}$$

only odd terms

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{[n!]^2} \left(\frac{x}{2}\right)^{2n}$$

only even

Recall that J_0 is like a damped cosine (even)

& J_1 is like a damped sine (odd)

Recall that $\frac{d}{dx}\cos x = -\sin x$

Similarly, $\frac{d}{dx} J_0(x) = -J_1(x)$

Proof: $J_0'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{[n!]^2} 2n\left(\frac{x}{2}\right)^{2n-1}\frac{1}{2}$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n!\,(n-1)!}\left(\frac{x}{2}\right)^{2n-1}$$

Re-index

$m = n-1 \Rightarrow n = m+1$

$$\Rightarrow \quad = \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{(m+1)!\,m!}\left(\frac{x}{2}\right)^{2m+1}$$

$$= (-1)\cdot \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)!\,m!}\left(\frac{x}{2}\right)^{2m+1} = -J_1(x)$$

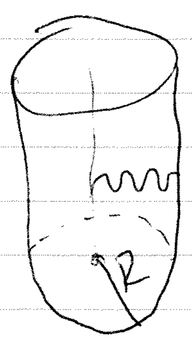
Approximating Bessel functions

In many applications, we are interested in $J_{\pm p}(x)$ where

$x = kR = \frac{2\pi}{\lambda} R$ where

λ is the wavelength &

R is the radius of the cylinder. For example, λ could be a sound or EM wave in a cylindrical cavity like an organ pipe or a cylindrical wave guide



- We have resonance when

$$x = kR = \frac{2\pi}{\lambda} R \approx 1$$

That occurs when $R \approx \lambda$ typically

There are two extremes:

$\lambda \ll R \Rightarrow x = kR \gg 1$ many resonances

$\lambda \gg R \Rightarrow x = kR \ll 1$ no resonances

For $\lambda \gg R \Rightarrow x \ll 1$,

we just keep the 1st term in the series. Let $p = 0, 1, 2, \ldots$

$$J_p(x) \underset{|x| \ll 1}{\sim} \frac{1}{p!}\left(\frac{x}{2}\right)^p + O[x^{p+2}]$$

No resonances.

- In the opposite extreme, expand $J_p(x)$ in an asymptotic series about $x = \infty$ (Mathematical)

$$J_p(x) \underset{|x| \gg 1}{\sim} \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{p\pi}{2} - \frac{\pi}{4}\right)$$

resonances but Amplitude drops as $\sqrt{1/x}$