Lecture 11

Asymptotic Series

There are divergent series that can be of practical use.

Consider

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$$

$$= \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt$$

We will expand in inverse powers of x, in order to characterize the behavior of $\operatorname{erfc}(x)$ for large x (i.e., $|x| \gg 1$).

Consider that

$$e^{-t^2} = \frac{1}{t} \cdot t e^{-t^2} = \frac{1}{t} \frac{d}{dt}\left(-\frac{1}{2} e^{-t^2}\right)$$

& use this to integrate by parts

$$\int_x^\infty e^{-t^2} dt = \int_x^\infty \frac{1}{t} \frac{d}{dt}\left(-\frac{1}{2} e^{-t^2}\right) dt$$

$$= \frac{1}{t}\left(-\frac{1}{2} e^{-t^2}\right)\Big|_x^\infty$$

$$- \int_x^\infty \left(-\frac{1}{2} e^{-t^2}\right)\left(-\frac{1}{t^2}\right) dt$$

$$= \frac{1}{2x} e^{-x^2} - \frac{1}{2} \int_x^\infty \frac{1}{t^2} e^{-t^2} dt$$

For the last integral, use a similar trick:

$$\frac{1}{t^2} e^{-t^2} = \frac{1}{t^3} \frac{d}{dt}\left(-\frac{1}{2} e^{-t^2}\right)$$

Again integrate by parts:

$$\int_x^\infty \frac{1}{t^2} e^{-t^2}\, dt = \int_x^\infty \frac{d}{dt}\left(-\frac{1}{2} e^{-t^2}\right)\cdot \frac{1}{t^3}\, dt$$

$$= \frac{1}{t^3}\left(-\frac{1}{2} e^{-t^2}\right)\Big|_x^\infty$$

$$- \int_x^\infty \left(-\frac{1}{2} e^{-t^2}\right)\left(-\frac{3}{t^4}\right) dt$$

$$= \frac{1}{2x^3} e^{-x^2} - \frac{3}{2}\int_x^\infty \frac{1}{t^4} e^{-t^2}\, dt$$

~~can continue~~

All of this gives

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt$$

$$= \frac{e^{-x^2}}{x\sqrt{\pi}}\left(1 - \frac{1}{2x^2}\right) + \frac{3}{2\sqrt{\pi}} \int_x^\infty \frac{1}{t^4} e^{-t^2}\, dt$$

can continue this chain to find that

$$\mathrm{erfc}(x) \sim \frac{e^{-x^2}}{x\sqrt{\pi}}\left(1 - \frac{1}{2x^2} + \frac{1\cdot 3}{(2x^2)^2} - \frac{1\cdot 3\cdot 5}{(2x^2)^3} + \ldots\right)$$

$\boxed{\text{Bring up Mathematica}}$ ...

The above series diverges for every x, because the factors in the numerator are growing fast compared to polynomial growth of x in the denominator.

However, the expression

$$\mathrm{erf}_c(x) = \frac{e^{-x^2}}{x\sqrt{\pi}}\left(1 - \frac{1}{2x^2}\right) + \frac{3}{2\sqrt{\pi}}\int_x^\infty \frac{1}{t^4} e^{-t^2}\, dt$$

is <u>exact</u>. It also thus converges.

However, for exact integral, we can prove that the final term is negligible. Can then use

$$\frac{e^{-x^2}}{x\sqrt{\pi}}\left(1-\frac{1}{2x^2}\right)$$ as a good

approximation of erfc(x) for large x.

- Let us estimate the size of

$$\int_x^\infty \frac{1}{t^4} e^{-t^2}\,dt$$ for large x.

t goes from x to ∞ &

so we have that $x \le t$

& $1/x \ge 1/t$

can rewrite integral as

$$\int_x^\infty t^{-4} e^{-t^2}\,dt = \int_x^\infty \frac{1}{t^5}\, t e^{-t^2}\,dt$$

$$\Rightarrow \int_x^\infty t^{-4} e^{-t^2} \le \int_x^\infty \frac{1}{x^5} t e^{-t^2} dt$$

$$= \frac{1}{x^5} \int_x^\infty t e^{-t^2} dt = \frac{1}{x^5} \left(-\frac{1}{2} e^{-t^2} \right) \Big|_x^\infty$$

$$= \frac{e^{-x^2}}{2x^5}$$

$$\Rightarrow \operatorname{erfc}(x) \le \frac{e^{-x^2}}{x\sqrt{\pi}} \left(1 - \frac{1}{2x^2} + \frac{1}{(2x)^4} \right)$$

Evaluation implies that when we stop w/ the term ~~in~~ e^{-x^2}/x^3 the error is of the order of e^{-x^2}/x^5, which is much smaller than e^{-x^2}/x^3 as x increases.

$\Rightarrow$ 1st two terms give a good approximation when $x \gg 1$.

If we stop @ $\frac{e^{-x^2}}{x^3}$, ratio of error to last term kept is

$$\frac{e^{-x^2}/x^5}{e^{-x^2}/x^3} = \frac{x^3}{x^5} = \frac{1}{x^2},$$

which tends to zero as $x \to \infty$.

"Error" in asymptotic expansion means difference between function being expanded & a partial sum (1st N terms) of the series

A series is called an asymptotic series (about ∞) of a function $f(x)$, if, for each fixed N, ratio of error to last term kept $\to 0$ as $x \to \infty$. That is

$$\underbrace{f(x) \sim \sum_{n=0}^{\infty} \phi_n(x)}$$

"$\sum_{n=0}^{\infty} \phi_n$ is an asymptotic expansion of $f(x)$"

if for each fixed N

$$\frac{\left| f(x) - \sum_{n=0}^{N} \phi_n(x) \right|}{\phi_N(x)} \to 0 \text{ as } x \to \infty$$

Often, the terms of an asymptotic series are inverse powers of x.

Then the above can be written as

$$f(x) \sim \sum_{n=0}^{\infty} \frac{a_n}{x^n}$$ if for each fixed N

$$\left| f(x) - \sum_{n=0}^{N} \frac{a_n}{x^n} \right| \cdot x^N \to 0 \text{ as } x \to \infty$$

An alternative way of deriving an asymptotic expansion is to substitute $x = \frac{1}{y}$ & then Taylor expand about $y = 0$.

An asymptotic expansion need not always diverge. To test for convergence @ x, fix x & let $n \to \infty$.

To test if series is asymptotic, fix n & let x tend to a limit (such as ∞).

A given series may pass both tests, only one, or neither.

Stirling's Formula

can be used to simplify formulas involving factorial or Γ function.

$$n! \sim n^n e^{-n} \sqrt{2\pi n}$$

$$\Gamma(p+1) \sim p^p e^{-p} \sqrt{2\pi p}$$

"$\sim$" means "is asymptotic to"

& the ratio $\dfrac{n!}{n^n e^{-n} \sqrt{2\pi n}} \to 1$

as $n \to \infty$.

Absolute error increases (diff between actual value & approximation) but relative error (ratio of error to $n!$) decreases as $n \to \infty$.

Give some idea now of how this approximation comes about

$$\Gamma(p+1) = p! = \int_0^{\infty} x^p e^{-x} dx$$

$$= \int_0^{\infty} e^{p \ln x - x} dx$$

Set y to be such that

$x = p + y\sqrt{p}$, so that

$dx = \sqrt{p}\, dy$ &

$x = 0$ corresponds to $y = -\sqrt{p}$

then the above becomes

$$\Gamma(p+1) = \int_{-\sqrt{p}}^{\infty} e^{p \ln(p + y\sqrt{p}) - p - y\sqrt{p}} \sqrt{p}\, dy$$

For large p, expand logarithm as

$$\ln(p + y\sqrt{p}) = \ln p + \ln\left(1 + \frac{y}{\sqrt{p}}\right) =$$

$= \ln p + \frac{y}{\sqrt{p}} - \frac{y^2}{2p} + \ldots$

Substitute to get

$$\Gamma(p+1) \sim \int_{-\sqrt{p}}^{\infty} e^{p\ln p + y\sqrt{p} - \frac{y^2}{2} - p - y\sqrt{p}} \cdot \sqrt{p}\,dy$$

$$= e^{p\ln p - p}\sqrt{p}\int_{-\sqrt{p}}^{\infty} e^{-y^2/2}\,dy$$

$$= e^{p\ln p - p} \cdot \sqrt{p}\left[\int_{-\infty}^{\infty} e^{-y^2/2}\,dy - \int_{-\infty}^{-\sqrt{p}} e^{-y^2/2}\,dy\right.$$

1st integral is $\sqrt{2\pi}$

Second tends to zero as p gets large,

so that $\Gamma(p+1) \sim p^p e^{-p}\sqrt{2\pi p}$

With more work, can give an asymptotic expansion as

$$\Gamma(p+1) = p^p e^{-p} \sqrt{2\pi p}\left(1 + \frac{1}{12p} + \frac{1}{288p^2} + \ldots\right)$$