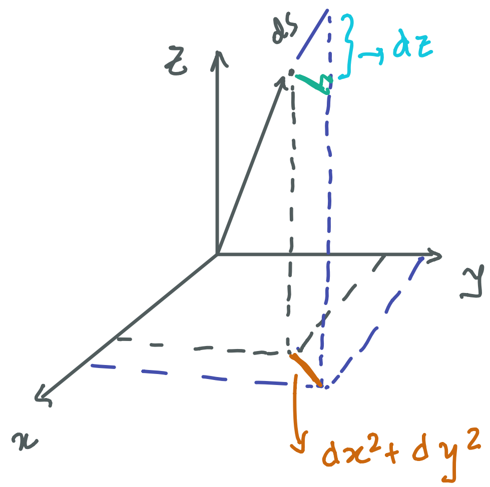
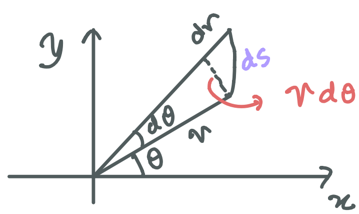
Cartesian coordinates (x, y, z)



$$\boxed{ds^2 = dx^2 + dy^2 + dz^2}$$

Curvilinear coordinates: cylindrical coordinates (r, θ, z)

first, let's restrict ourselves to x-y plane



ds ?

$$ds^2 = dr^2 + (r\,d\theta)^2$$

$$\boxed{ds^2 = dr^2 + r^2 d\theta^2}$$

arc length

In this particular example, it was very straightforward to find ds. Our goal is to develop a framework to find ds for any coordinate system.

For cylindrical coordinate system, one can write the following relations

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

Furthermore, $dx = dr\cos\theta - r\sin\theta\, d\theta$

$$dy = dr\sin\theta + r\cos\theta\, d\theta$$

$$dz = dz$$

now $ds^2 = dx^2 + dy^2 + dz^2 = [dr\cos\theta - r\sin\theta\, d\theta]^2 + [dr\sin\theta + r\cos\theta\, d\theta]^2 + dz^2$

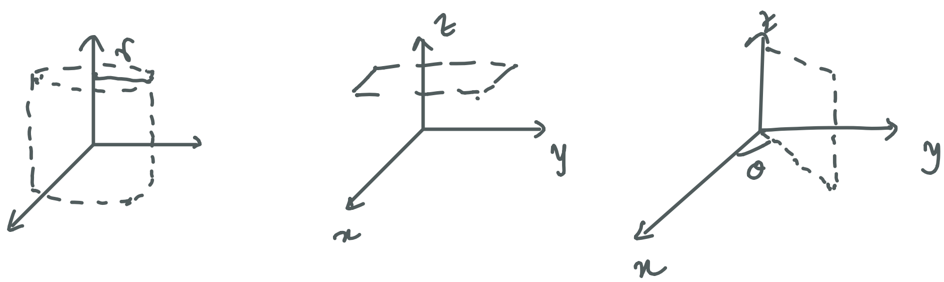
$$= dr^2 + r^2 d\theta^2 + dz^2$$

note: no cross terms survives in the cylindrical coordinate system.

When such a thing happens, we call the coordinate system orthogonal.

Let's analyze the coordinate surfaces:

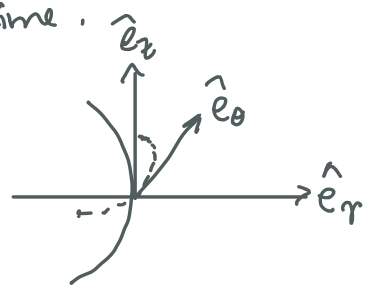
1) $r = \text{const}$ 2) $z = \text{const}$ 3) $\theta = \text{constant}$



these surface are pendicular to each other at every point they intersect.

Coordinate lines or directions.

now you can also talk about the intersection of two surfaces at a time.



We will call any coordinate system curvilinear when the coordinate surfaces are not planes (or some of them) and the coordinate lines are curves rather than straight line.

Scale factors and Basis vectors.

In rectangular system $x \to x + dx$; $y, z =$ constant

then particle moves a distance dx

however, in a cylindrical coordinate system if θ changes by the amount $d\theta$ then $\boxed{ds = \underbrace{r}_{\text{Scale factor}} d\theta}$

How to figure out scale factors?

If the transformation is orthogonal, as it was the case with cylindrical coordinates then one can find the scale factors from ds^2

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2$$

Scale factors $1, r, 1$

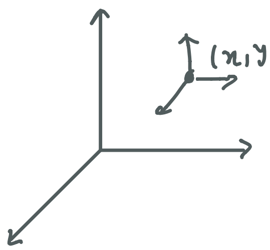
Let's write a vector in cylindrical coordinate system

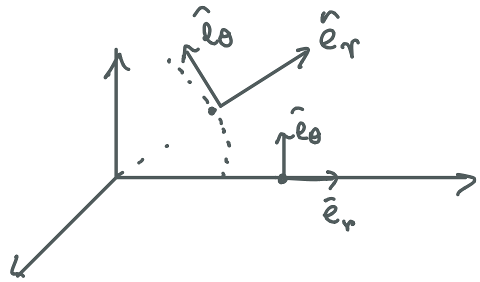
$$d\vec{s} = \hat{e}_r\, dr + \hat{e}_\theta\, r\, d\theta + \hat{e}_z\, dz$$

$d\vec{s} \cdot d\vec{s} = ds^2$ $\quad \because \hat{e}_r, \hat{e}_\theta, \hat{e}_z$ are orthonormal to each other.

we now discuss an important distinction b/w rectangular coordinate system and curvilinear coordinate system.

$\hat{i}, \hat{j}, \hat{k}$ are constant in magnitude and direction, but that's not the case with $\hat{e}_r, \hat{e}_\theta$.





these basis vectors also change if you go from one point to another

so there should be a well defined method to connect these two coordinate systems.

$$ds = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$
$$= (\cos\theta\, dr - r\sin\theta\, d\theta)\,\hat{i} + (\sin\theta\, dr + r\cos\theta\, d\theta)\,\hat{j} + dz\,\hat{k}$$
$$= (\hat{i}\cos\theta + \hat{j}\sin\theta)\,dr + [-\hat{i}\, r\sin\theta + \hat{j}\, r\cos\theta]\,\hat{j} + dz\,\hat{k}$$

$$\left.\begin{aligned} \hat{e}_r &= \hat{i}\cos\theta + \hat{j}\sin\theta \\ r\,\hat{e}_\theta &= -\hat{i}\, r\sin\theta + \hat{j}\, r\cos\theta \\ \hat{e}_z &= \hat{k} \end{aligned}\right\}$$

* all of these are unit vectors.
* it is easy to see how they change directions on every point.

Let's see what happens when we take time derivative of displacement in cylindrical coordinates:

$$\vec{s} = r\,\hat{e}_r + z\,\hat{e}_z$$

$$\frac{d\vec{s}}{dt} = \frac{dr}{dt}\hat{e}_r + r\frac{d\hat{e}_r}{dt} + \frac{dz}{dt}\hat{e}_z$$

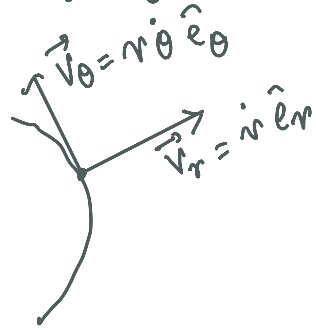
$$= \dot{r}\,\hat{e}_r + r\frac{d\hat{e}_r}{dt} + \dot{z}\,\hat{e}_z$$

$$\frac{d\hat{e}_r}{dt} = ? \quad = \quad \hat{i}\,(-\sin\theta)\frac{d\theta}{dt} + \hat{j}\cos\theta\frac{d\theta}{dt}$$

$$= \dot{\theta}\,\hat{e}_\theta$$

$$\boxed{\dot{\vec{s}} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta + \dot{z}\,\hat{e}_z}$$

Let's just focus on the motion in a plane.



What if we differentiate two times?

$$\vec{a} = \dot{\vec{v}} = \ddot{r}\,\hat{e}_r + \dot{r}\,\dot{\hat{e}}_r + \dot{r}\dot{\theta}\,\hat{e}_\theta + r\ddot{\theta}\,\hat{e}_\theta + r\dot{\theta}\,\dot{\hat{e}}_\theta + \ddot{z}\,\hat{e}_z$$

$$= \ddot{r}\,\hat{e}_r + \dot{r}\dot{\theta}\,\hat{e}_\theta + \dot{r}\dot{\theta}\,\hat{e}_\theta + r\ddot{\theta}\,\hat{e}_\theta + r\dot{\theta}^2(-\hat{e}_r) + \ddot{z}\,\hat{e}_z$$

$$= (\ddot{r} - r\dot{\theta}^2)\,\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\hat{e}_\theta + \ddot{z}\,\hat{e}_z$$

$\downarrow$

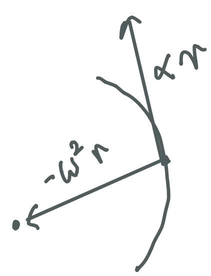
even if r was not changing with time, we know there is a radial force/accelaration acting on particle.

$$\left[\vec{a}_r = -\omega^2 r\,\hat{e}_r\right]$$

$$|\vec{a}| = \sqrt{a_r^2 + a_\theta^2 + a_z^2}$$

moreover, if $r \neq f(t)$ then $\vec{a}_\theta = r\ddot{\theta}\,\hat{e}_\theta$

$\vec{a}_\theta = r\alpha\,\hat{e}_\theta$



Example 2: In our example, we described a coordinate system with variables $x_1 = r$, $x_2 = \theta$, $x_3 = z$. Is there any formal way of describing a coordinate system with arbitrary x_1, x_2, x_3.

$$d\vec{s} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$$= \frac{\partial x}{\partial x_n} dx_n\,\hat{i} + \frac{\partial y}{\partial x_n} dx_n\,\hat{j} + \frac{\partial z}{\partial x_n} dx_n\,\hat{k}$$

$$= \left(\frac{\partial x}{\partial x_1}\hat{i} + \frac{\partial y}{\partial x_1}\hat{j} + \frac{\partial z}{\partial x_1}\hat{k}\right) dx_1 + \left(\frac{\partial x}{\partial x_2}\hat{i} + \frac{\partial y}{\partial x_2}\hat{j} + \frac{\partial z}{\partial x_2}\hat{k}\right) dx_2 + \cdots$$

$$= a_1\,dx_1 + a_2\,dx_2 + a_3\,dx_3$$

where $\boxed{\vec{a}_n = \frac{\partial \vec{s}}{\partial x_n}}$

Define $g_{ij} = \vec{a}_i \cdot \vec{a}_j$ $\qquad ds^2 = d\vec{s}\cdot d\vec{s}$

$$d\vec{s}\cdot d\vec{s} = \vec{a}_1\cdot\vec{a}_1\,dx_1^2 + \vec{a}_2\cdot\vec{a}_2\,dx_2^2 + \vec{a}_3\cdot\vec{a}_3\,dx_3^2$$
$$+ 2\vec{a}_1\cdot\vec{a}_2\,dx_1\,dx_2 + 2\vec{a}_1\cdot\vec{a}_3\,dx_1\,dx_3 + 2\vec{a}_2\cdot\vec{a}_3\,dx_2\,dx_3$$

$$\equiv (dx_1\ dx_2\ dx_3)\begin{bmatrix} \vec{a}_1\cdot\vec{a}_1 & \vec{a}_1\cdot\vec{a}_2 & \vec{a}_1\cdot\vec{a}_3 \\ \vec{a}_2\cdot\vec{a}_1 & \vec{a}_2\cdot\vec{a}_2 & \vec{a}_2\cdot\vec{a}_3 \\ \vec{a}_3\cdot\vec{a}_1 & \vec{a}_3\cdot\vec{a}_2 & \vec{a}_3\cdot\vec{a}_3 \end{bmatrix}\begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}$$

$$\downarrow$$

$$\equiv \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix}$$

$$\Rightarrow \underbrace{ds^2 = \underbrace{g_{ij}}_{\text{components of the metric tensor.}}\,dx_i\,dx_j}$$

that's the expression you also see in relativity.

if the basis vectors are orthogonal, then $a_i \cdot a_j = 0$ for $i \neq j$

$$\Rightarrow \quad ds^2 = (dx_1\ dx_2\ dx_3)\begin{pmatrix} h_1^2 & 0 & 0 \\ 0 & h_2^2 & 0 \\ 0 & 0 & h_3^2 \end{pmatrix}\begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}$$

$$d\vec{s} = h_1 dx_1 \hat{e}_1 + h_2 dx_2 \hat{e}_2 + h_3 dx_3 \hat{e}_3$$

for cylindrical $h_1 = 1,\ h_2 = r,\ h_3 = 1$

for spherical $h_1 = 1,\ h_2 = r \quad h_3 = r\sin\theta$
