Lecture 3

Inertia tensor

In mechanics, if there is rotation about a fixed axis, then

torque $\vec{\tau}$,

angular momentum $\vec{L}$, &

angular velocity $\vec{\omega}$

are all parallel & related by

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$ (angular version of $\vec{F} = \frac{d\vec{p}}{dt}$)

$$\vec{L} = I\vec{\omega}$$ ($\vec{p} = m\vec{v}$)

where I is moment of inertia

If I does not change in time,

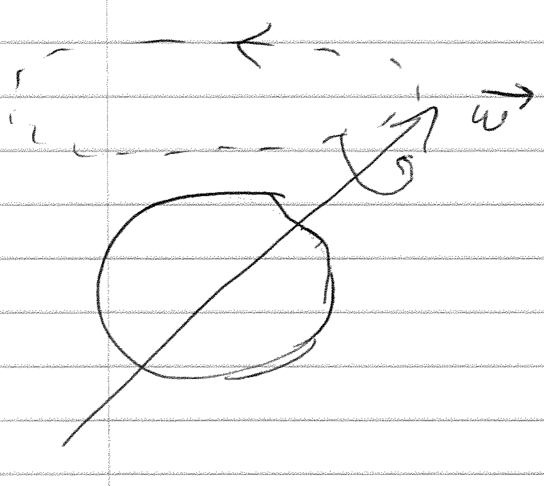
then $\vec{\tau} = I\frac{d\vec{\omega}}{dt}$ ($\vec{F} = m\frac{d\vec{v}}{dt} = m\vec{a}$)

- For rotation about a fixed axis, I is a rank zero tensor (scalar) & $\vec{\tau} \parallel \vec{\omega} \parallel \vec{L}$

- But if the rotation axis is <u>not fixed</u>, then $\overleftrightarrow{I}$ is a second rank tensor & in general $\vec{\tau}$, $\vec{L}$, & $\vec{\omega}$ are not parallel.

<u>Example</u>

Earth is not a perfect sphere



like a spinning top, its axis of rotation precesses w/ a period of once every 25000 years.

③

In this case, its angular momentum $\vec{L}$ is related to its angular velocity $\vec{\omega}$ by the inertia tensor $\overleftrightarrow{I}$, which is a rank-two tensor.

$$\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega}$$

where $\overleftrightarrow{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$

$$= \begin{bmatrix} I_{11} & I_{12} & \cdots \\ I_{21} & \cdot & \\ \vdots & & I_{33} \end{bmatrix}$$

In component form,

$$L_j = \sum_k I_{jk} \omega_k \equiv I_{jk} \omega_k$$

(Einstein)

If earth was a perfect sphere, then we recover the well known formula

$$I_0 = \frac{2}{5} M R^2 \quad \&$$

$$\overleftrightarrow{I} = I_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

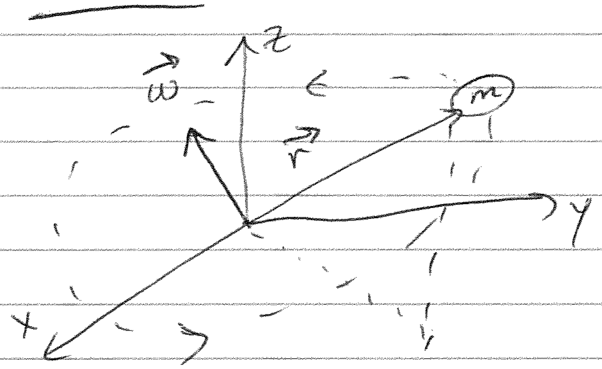
The off-diagonal terms I_{xy} & non-equal diagonal terms measure "non-spherical-ness" of a given body.

Fundamental example of a point mass

What are the components of the inertia tensor in this case?

Let m be a point mass @ $\vec{r} = (x, y, z)$ w/ angular velocity $\vec{\omega}$ about the origin.

Picture:



Recall from mechanics that

$$\vec{L} = m\,\vec{r} \times (\vec{\omega} \times \vec{r}),$$

so that $\vec{L} \not\parallel \vec{\omega}$ in general.

Then use the relations

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$$

$$\Rightarrow \quad m\,\vec{r} \times (\vec{\omega} \times \vec{r}) = (m\,\vec{r} \cdot \vec{r})\vec{\omega} - (m\,\vec{r} \cdot \vec{\omega})\vec{r}$$

$$\Rightarrow \quad \vec{L} = m\left[r^2 \vec{\omega} - (\vec{r} \cdot \vec{\omega})\vec{r}\right]$$

From this, observe that

$$\vec{L} \parallel \vec{\omega} \iff \vec{r} \cdot \vec{\omega} = 0$$

meaning that axis of rotation in this case does not change w/ changing $\vec{r}$.

Using Einstein notation,
we can write this as

$$L_i = m\left[x_j \overbrace{x_j}^{\text{dummy}} \omega_i - x_k \overbrace{\omega_k}^{\text{dummy}} x_i \right]$$

$$\underbrace{x_j x_j}_{\vec{r}\cdot\vec{r}} \qquad \underbrace{x_k \omega_k}_{\vec{r}\cdot\vec{\omega}}$$

index

$$= m\left[x_j x_j \omega_i - x_k x_i \omega_k \right]$$

OR

$$L_x = m\left\{ r^2 \omega_x - (x\omega_x + y\omega_y + z\omega_z)x \right\}$$
$$L_y = m\left\{ r^2 \omega_y - (x\omega_x + y\omega_y + z\omega_z)y \right\}$$
$$L_z = m\left\{ r^2 \omega_z - (x\omega_x + y\omega_y + z\omega_z)z \right\}$$

OR

$$L_x = m\left[(r^2 - x^2)\omega_x - xy\omega_y - zx\omega_z \right]$$
$$L_y = m\left\{ (r^2 - y^2)\omega_y - xy\omega_x - yz\omega_z \right\}$$
$$L_z = m\left[(r^2 - z^2)\omega_z - zx\omega_x - yz\omega_y \right]$$

OR

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} m(r^2-x^2) & -mxy & -mxz \\ -mxy & m(r^2-y^2) & -myz \\ -mxz & -myz & m(r^2-z^2) \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$\Rightarrow \quad \vec{L} = \overleftrightarrow{I} \cdot \vec{\omega} \quad \text{w/}$$

$$\overleftrightarrow{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

$$= m \begin{bmatrix} r^2-x^2 & -xy & -xz \\ -xy & r^2-y^2 & -yz \\ -xz & -yz & r^2-z^2 \end{bmatrix}$$

So this is the moment of inertia tensor

It is symmetric by inspection

b/c $I_{ij} = I_{ji} \ \forall i, j$

Using the notation

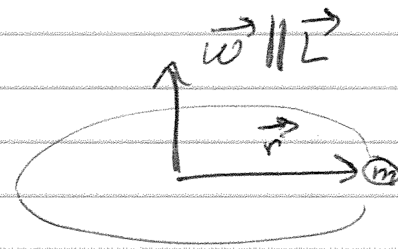
$$\overleftrightarrow{\mathbb{1}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \&$$

$$\overleftrightarrow{J} \quad w/ \quad \{J_{ij}\} = \{x_i x_j\}$$

$$= \begin{bmatrix} x_1x_1 & x_1x_2 & x_1x_3 \\ x_2x_1 & x_2x_2 & x_2x_3 \\ x_3x_1 & x_3x_2 & x_3x_3 \end{bmatrix}$$

we can write

$$\overleftrightarrow{I} = mr^2 \overleftrightarrow{\mathbb{1}} - m \overleftrightarrow{J}$$



If we had chosen the plane of rotation in x-y plane, then the situation would simplify tremendously b/c

$$\vec{r} \perp \vec{\omega} \Rightarrow \vec{r}\cdot\vec{\omega} = 0$$

so that

$$\vec{L} = m\left[r^2\vec{\omega} - \cancelto{0}{(\vec{r}\cdot\vec{\omega})}\,\vec{r}\right]$$

$$= mr^2\vec{\omega} = I\vec{\omega}$$

where $I = mr^2$ is the scalar moment of inertia.

This would be the ideal choice for a point mass, but for a mass distribution, there is no obvious choice of coordinates that turns $\overleftrightarrow{I}$ into a scalar.

However, we can always find principal axes of moments.

Noting that

$$r^2 - x^2 = y^2 + z^2$$

$$r^2 - y^2 = x^2 + z^2$$

$$r^2 - z^2 = x^2 + y^2$$

We can write

$$\overleftrightarrow{I} = m \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix}$$

or

$$I_{ij} = mr^2 \delta_{ij} - x_i x_j$$

where $\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

Any 3x3 symmetric matrix can be diagonalized by switching from x, y, z to x', y', z' in which

$$\overleftrightarrow{I} = \begin{bmatrix} I_{x'x'} & 0 & 0 \\ 0 & I_{y'y'} & 0 \\ 0 & 0 & I_{z'z'} \end{bmatrix}$$

Suppose that $\rho(\vec{r})\, dr^3 = dm$
is a continuous mass distribution over volume V.

We can construct $\overleftrightarrow{I}$

by integrating

$$I_{xx} = \int_V dm\, (y^2+z^2) = \int_V dr^3 \rho(\vec{r})(y^2+z^2)$$

$$I_{yy} = \int_V dr^3 \rho(\vec{r})\, (x^2+z^2)$$

$$I_{zz} = \int_V dr^3 \rho(\vec{r})\, (x^2+y^2)$$

$$I_{xy} = -\int_V dr^3\, \rho(\vec{r})\, xy$$

$$I_{xz} = -\int_V dr^3 \rho(\vec{r})\, xz$$

$$I_{yz} = -\int_V dr^3\, \rho(\vec{r})\, yz$$

for a discrete mass distribution,
it would be

$$I_{xx} = \sum_i m_i (y_i^2 + z_i^2)$$

$$I_{yy} = \sum_i m_i (x_i^2 + z_i^2)$$

etc. . . .

Finding principal moments of axes of inertia

$$\overleftrightarrow{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix}$$

can be diagonalized.

Goal is to find a new coordinate system (x', y', z') such that

$$I'_{ij} = \begin{bmatrix} I_{x'x'} & 0 & 0 \\ 0 & I_{y'y'} & 0 \\ 0 & 0 & I_{z'z'} \end{bmatrix}$$

these are the <u>principal moments</u> of inertia &

(x', y', z') are the <u>principal axes</u>

generalizes notion of
center of mass