Lecture 14

Continuing w/ semidefinite programming

If α is ∞ or $-\infty$, then there is no optimal primal solution & same for when β is ∞ or $-\infty$ (no optimal dual solution)

Even when α & β are finite, there may not be an optimal solution

Example: Suppose 2-D spaces $\mathcal{X}$ & $\mathcal{Y}$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\Phi(X) = \begin{bmatrix} 0 & X_{1,2} \\ X_{2,1} & 0 \end{bmatrix}$$

can show that $\alpha = 0$ but there is no optimal solution

Suppose that $X \in PSD(\mathcal{X})$ is primal feasible

$$\Phi(X) = B \Rightarrow X = \begin{bmatrix} X_{1,1} & 1 \\ 1 & X_{2,2} \end{bmatrix}$$

In order for X to be PSD,

~~we~~ we need $X_{1,1} \geq 0$ & $X_{2,2} \geq 0$

because $e_1^t X e_1 = X_{1,1}$ where $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$e_2^t X e_2 = X_{2,2}$ " $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

the determinant is $X_{1,1} X_{2,2} - 1$

& ~~we~~ we need this ≥ 0

$$\Rightarrow X_{1,1} \cdot X_{2,2} \geq 1$$

In order for this to be true, we need that $X_{1,1} > 0$ (couldn't be satisfied if $X_{1,1} = 0$)

But this means that the objective function

$$\langle A, X \rangle = Tr\left\{ \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_{1,1} & 1 \\ 1 & X_{2,2} \end{bmatrix} \right\} = -X_{1,1} < 0$$

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But we could pick

$$X_n = \begin{bmatrix} 1/n & 1 \\ 1 & n \end{bmatrix}$$ for each positive integer n

So $X_n \in A$ (primal feasible set)

& $\langle A, X_n \rangle = -1/n$

so $\alpha > -1/n$ $\forall$ positive n

$\Rightarrow$ $\alpha = 0$ (supremum)

But there is no $X \in A$ that achieves this.

Every semidefinite program obeys weak duality, which is the statement that $\alpha \le \beta$.

Proof:

trivial if $\mathcal{A} = \phi$ b/c then $\alpha = -\infty$

or if $\mathcal{B} = \phi$ b/c then $\beta = \infty$

So suppose $\mathcal{A} \neq \phi$ + $\mathcal{B} \neq \phi$

For primal feasible $X \in \mathcal{A}$ + dual feasible $Y \in \mathcal{B}$ we have

$$\langle A, X\rangle \le \langle \Phi^*(Y), X\rangle \quad \text{(from dual constraint)}$$
$$= \langle Y, \Phi(X)\rangle \quad \text{(def'n of adjoint)}$$
$$= \langle Y, B\rangle \quad \text{(from primal constraint)}$$

This holds for all feasible X + Y

$$\Rightarrow \sup_X \langle A, X\rangle \le \inf_Y \langle Y, B\rangle \Rightarrow \alpha \le \beta$$

Weak duality implies that every dual feasible $Y \in \mathcal{B}$ gives an upper bound of $\langle Y, B \rangle$ on α & every primal feasible $X \in \mathcal{A}$ gives a lower bound on β

i.e.,
$$\langle A, X \rangle \leq \alpha \leq \beta \leq \langle B, Y \rangle$$

<u>Strong duality</u> is the condition that
$$\alpha = \beta$$

This is when semidefinite programming can be a very helpful analytical tool.

It doesn't always hold, as the following counter example demonstrates

Let $\mathcal{X} = \mathbb{C}^3$, $\mathcal{Y} = \mathbb{C}^2$

Set $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$\Phi(X) = \begin{bmatrix} X_{1,1} + X_{2,3} + X_{3,2} & 0 \\ 0 & X_{2,2} \end{bmatrix}$

$\Rightarrow$ PRIMAL IS to

maximize $-X_{1,1}$

subject to $X_{1,1} + X_{2,3} + X_{3,2} = 1$

$X_{2,2} = 0$ & $X \geq 0$

constraint $X(2,2) = 0$ & $X \geq 0$

imply that $X_{2,3} + X_{3,2} = 0$

Why? consider all

$\begin{bmatrix} X_{1,1} & X_{1,2} & X_{1,3} \\ X_{2,1} & X_{2,2} & X_{2,3} \\ X_{3,1} & X_{3,2} & X_{3,3} \end{bmatrix}$

submatrices must be positive semidefinite so this means

$0 = X_{2,3} = X_{3,2} = X_{1,2}$
$= X_{2,1}$

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So this means that $X_{1,1} = 1$

& thus $\alpha \leq -1$

can see that $\alpha = -1$ by

picking $X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

What is the dual problem?

1st figure out the adjoint map

$$\mathrm{Tr}\{ Y \, \Phi(X) \} = \mathrm{Tr}\{ \Phi^{+}(Y) \, X \}$$

$$Y = \begin{bmatrix} Y_{1,1} & Y_{1,2} \\ Y_{2,1} & Y_{2,2} \end{bmatrix} \qquad \Phi(X) = \begin{bmatrix} X_{1,1} + X_{2,3} + X_{3,2} & 0 \\ 0 & X_{2,2} \end{bmatrix}$$

$$\mathrm{Tr}\{ Y \cdot \Phi(X) \} = Y_{1,1} \left(X_{1,1} + X_{2,3} + X_{3,2} \right) + Y_{2,2} X_{2,2}$$

If we pick

$$\Phi^+(Y) = \begin{bmatrix} Y_{1,1} & 0 & 0 \\ 0 & Y_{2,2} & Y_{1,1} \\ 0 & Y_{1,1} & 0 \end{bmatrix}$$

then $\Phi^+(Y)\, X$ =

$$\mathrm{Tr}\left\{ \begin{bmatrix} Y_{1,1} & 0 & 0 \\ 0 & Y_{2,2} & Y_{1,1} \\ 0 & Y_{1,1} & 0 \end{bmatrix} \begin{bmatrix} X_{1,1} & X_{1,2} & X_{1,3} \\ X_{2,1} & X_{2,2} & X_{2,3} \\ X_{3,1} & X_{3,2} & X_{3,3} \end{bmatrix} \right\}$$

$$= Y_{1,1}X_{1,1} + Y_{2,2}X_{2,2} + Y_{1,1}X_{3,2} + Y_{1,1}X_{2,3}$$

match! so Φ^+ is as above

For the dual program, we have the constraint

$\Phi^+(Y) \geq A$ which is that

$$\begin{bmatrix} Y_{1,1}+1 & 0 & 0 \\ 0 & Y_{2,2} & Y_{1,1} \\ 0 & Y_{1,1} & 0 \end{bmatrix} \geq 0$$

but this means that

$\begin{bmatrix} Y_{2,2} & Y_{1,1} \\ Y_{1,1} & 0 \end{bmatrix} \geq 0$ & so $Y_{1,1} = 0$

But the dual value is

$\langle B, Y \rangle = 0$ so that

$\beta \geq 0$

we get that $\beta = 0$ by picking $Y = 0$

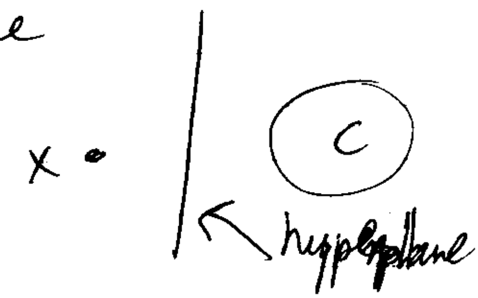
Strong duality: various conditions to check for when it holds.

<u>Slater's conditions are one set:</u>

1. If $\mathcal{A} \neq \phi$ & $\exists\, Y \in \text{Herm}(\mathcal{Y})$ such that $\Phi^{+}(Y) > A$, then $\alpha = \beta$ & there exists a primal feasible $X \in \mathcal{A}$ such that $\langle A, X \rangle = \alpha$

2. Similar condition for dual
   If $\mathcal{B} \neq \phi$ & $\exists\, X \in PSD(\mathcal{X})$ such that $\Phi(X) = B$ & $X > 0$
   then $\alpha = \beta$ & $\exists\, Y \in \mathcal{B}$ such that $\langle B, Y \rangle = \beta$

Important idea for proof:
separating hyperplane theorem –
every <u>closed</u> convex set C
can be separated from $x \notin C$
by a hyperplane



<u>Proof</u>: Let $K = \left\{ \begin{pmatrix} \Phi(X) & 0 \\ 0 & \langle A, X \rangle \end{pmatrix} : X \in \mathrm{PSD}(\mathcal{X}) \right\}$

state w/o proof

If $\exists$ $Y \in \mathrm{Herm}(\mathcal{Y})$ for which $\Phi^{\dagger}(Y) > A$ then K is a closed set.

Let $\varepsilon > 0$.

The matrix

$$(*) \begin{bmatrix} B & 0 \\ 0 & \alpha + \varepsilon \end{bmatrix} \text{ is not in } K$$

If it were, this would mean that there would exist $X \in \mathrm{PSD}(\mathcal{X})$ such that

$\Phi(X) = B$ & $\langle A, X\rangle > \alpha$

contradicting the optimality of α.

Since K is closed & convex, there exists a hyperplane that separates (*) from K. This hyperplane comes as a matrix $Y \in \mathrm{Herm}(\mathcal{Y})$ & a real λ such that

$$\langle Y, \Phi(X)\rangle + \lambda\langle A, X\rangle > \langle Y, B\rangle + \lambda(\alpha+\varepsilon)$$

(**) $$\forall X \in \mathrm{PSD}(\mathcal{X})$$

By assumption, $\mathcal{A}$ is nonempty, so that we can pick $X_0 \in \mathcal{A}$ for which $\Phi(X_0) = B$, implying

$$\lambda\langle A, X\rangle > \lambda(\alpha+\varepsilon)$$

But since $\alpha = \sup_X \langle A, X\rangle$, this implies that $\lambda < 0$

We can then divide (**) by $|\lambda|$ & thus we can choose $\lambda = -1$ WLOG.

& we get

$$\langle Y, \Phi(X)\rangle - \langle A, X\rangle > \langle Y, B\rangle - (\alpha+\varepsilon)$$

$$\Leftrightarrow \langle \Phi^{\dagger}(Y), X\rangle - \langle A, X\rangle > \langle Y, B\rangle - (\alpha+\varepsilon)$$

$$\Leftrightarrow \langle \Phi^{\dagger}(Y) - A, X\rangle > \langle Y, B\rangle - (\alpha+\varepsilon)$$

$$\forall X \in PSD(\mathcal{X})$$

quantity on right is independent of X. This implies that

$\Phi(Y) - A$ is PSD. If it were not, one could choose X to amplify a negative eigenvalue of $\Phi^{\dagger}(Y) - A$ & violate the inequality

So this $\Rightarrow$

$$\Phi^{\dagger}(Y) \geq A$$

& the Y is thus <u>dual feasible</u>

Set X in the last inequality to be 0

$\Rightarrow \quad \langle B, Y \rangle < \alpha + \varepsilon$

So we showed that $\forall \varepsilon > 0 \ \exists$ a dual feasible Y such that

$$\langle B, Y \rangle < \alpha + \varepsilon$$

$\Rightarrow \quad \beta = \inf_Y \langle B, Y \rangle < \alpha + \varepsilon$

Given that $\alpha \leq \beta$ from weak duality we have

$$\alpha \leq \beta < \alpha + \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, we get

$$\alpha = \beta$$

<u>Strong duality!</u>

Can also conclude that $\exists$ primal feasible X such that $\alpha = \langle A, X \rangle$