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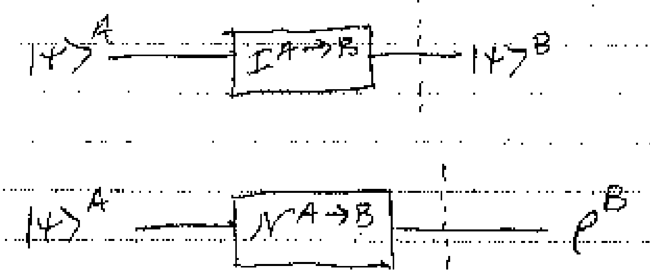
Distance Measures

- Protocols considered so far have been noiseless (though we developed noisy quantum theory)

- We would like ways of determining how close an imperfect protocol is to being perfect:



$D(\psi, \rho) \leq \epsilon$

↑ need a measure of distance

two main measures: trace distance and fidelity

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Trace distance arises from the trace norm of an operator M is

$$\|M\|_1 = \mathrm{Tr}\{\sqrt{M^\dagger M}\}$$

sum of singular values of M

just consider M Hermitian

then $M = \sum_i \mu_i |i\rangle\langle i|$

$\Rightarrow M^\dagger M = \sum_i |\mu_i|^2 |i\rangle\langle i|$

$\Rightarrow \sqrt{M^\dagger M} = \sum_i \sqrt{|\mu_i|^2}\, |i\rangle\langle i|$

$= \sum_i |\mu_i|\, |i\rangle\langle i|$

$\Rightarrow \mathrm{Tr}\{\sqrt{M^\dagger M}\} = \mathrm{Tr}\{\sum_i |\mu_i|\, |i\rangle\langle i|\}$

$= \sum_i |\mu_i|\, \mathrm{Tr}\{|i\rangle\langle i|\}$

$= \sum_i |\mu_i|$

absolute sum of eigen values

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5 useful properties for trace distance

1) positive definiteness: $\|M\|_1 \geq 0$

& $\|M\|_1 = 0 \iff M = 0$

2) Homogeneity: for $c \in \mathbb{C}$

$$\|cM\|_1 = |c| \, \|M\|_1$$

3) Triangle Inequality: (important for bounding errors)

$$\|M+N\|_1 \leq \|M\|_1 + \|N\|_1$$

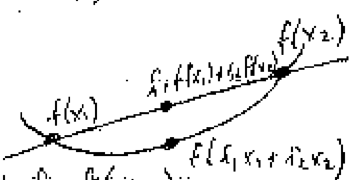
4) Isometric Invariance:

$$\|UMU^\dagger\|_1 = \|M\|_1 \quad \text{(why?)}$$

5) Convexity (holds for any norm):

$$\|\lambda_1 M + \lambda_2 N\|_1 \leq \lambda_1 \|M\|_1 + \lambda_2 \|N\|_1$$

where $\lambda_1, \lambda_2 \geq 0$ & $\lambda_1 + \lambda_2 = 1$



convex smile :)

$$f(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2)$$

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Trace norm induces trace distance:

$$\|M-N\|_1$$

useful as measure of distinguishability for density operators ρ & σ

ρ & σ are equal if $\|\rho-\sigma\|_1 = 0$

& an upper bound is

$$\|\rho-\sigma\|_1 = \|\rho + (-\sigma)\|_1$$
$$\leq \|\rho\|_1 + \|\sigma\|_1 = 1+1=2$$

Trace distance saturated when ρ & σ are on orthogonal subspaces

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Different characterization of trace distance:

$$\|\rho-\sigma\|_1 = 2 \max_{0 \leq \Lambda \leq I} \mathrm{Tr}\{\Lambda(\rho-\sigma)\}$$

↑ think of Λ as part of measurement used to distinguish ρ from σ (will see this later)

Proof:

$\rho-\sigma$ Hermitian

$$\Rightarrow \rho-\sigma = U D U^\dagger$$
$$= U(D^+ - D^-)U^\dagger$$
$$= \underbrace{U D^+ U^\dagger}_{\alpha^+} - \underbrace{U D^- U^\dagger}_{\alpha^-}$$

Let Π^+ be projector onto α^+ &

Π^- projector onto α^-

then

$$|\rho-\sigma| = \alpha^+ + \alpha^-$$

So $\|\rho-\sigma\|_1 = \mathrm{Tr}\{|\rho-\sigma|\}$

$$= \mathrm{Tr}\{\alpha^+ + \alpha^-\}$$
$$= \mathrm{Tr}\{\alpha^+\} + \mathrm{Tr}\{\alpha^-\}$$

But $Tr\{\alpha^+\} - Tr\{\alpha^-\}$

$= Tr\{\alpha^+ - \alpha^-\}$

$= Tr\{\rho - \sigma\} = 0$

$\therefore Tr\{\alpha^+\} = Tr\{\alpha^-\}$

So, $\|\rho - \sigma\|_1 = 2Tr\{\alpha^+\}$

Now consider

$$2Tr\{\Pi^+(\rho-\sigma)\} = 2Tr\{\Pi^+(\alpha^+ - \alpha^-)\}$$
$$= 2Tr\{\Pi^+\alpha^+\}$$
$$= 2Tr\{\alpha^+\}$$
$$= \|\rho-\sigma\|_1$$

prove that Π^+ is maximizing operator:

$$2Tr\{\Lambda(\rho-\sigma)\} = 2Tr\{\Lambda(\alpha^+ - \alpha^-)\}$$
$$\leq 2Tr\{\Lambda\alpha^+\}$$
$$\leq 2Tr\{\alpha^+\}$$
$$= \|\rho-\sigma\|_1$$

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Operational Interpretation of Trace Distance (sets it on firm footing)

Hypothesis testing – prepare ρ_0 or ρ_1 w/ equal probability $p_X(0) = p_X(1) = 1/2$

Bob doesn't know which one prepared & must perform a measurement to distinguish – POVM $\{\Lambda_0, \Lambda_1\}$

$\Lambda_0 \rightarrow$ guess state was ρ_0
$\Lambda_1 \rightarrow$ guess state was ρ_1

probability of error is sum of false positive & false negative probabilities

$$p_e = p(0|1)\,p_X(1) + p(1|0)\,p_X(0)$$
$$= \mathrm{Tr}\{\Lambda_0 \rho_1\}\,1/2 + \mathrm{Tr}\{\Lambda_1 \rho_0\}\,1/2$$

using the fact that $\Lambda_0 + \Lambda_1 = I$, can rewrite p_e

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$$p_e = \frac{1}{2}\left[\mathrm{Tr}\{\Lambda_0 \rho_1\} + \mathrm{Tr}\{\Lambda_1 \rho_0\}\right]$$
$$= \frac{1}{2}\left\{\mathrm{Tr}\{\Lambda_0 \rho_1\} + \mathrm{Tr}\{(I - \Lambda_0)\rho_0\}\right\}$$
$$= \frac{1}{2}\left\{\mathrm{Tr}\{\rho_0\} + \mathrm{Tr}\{\Lambda_0(\rho_1 - \rho_0)\}\right\}$$
$$= \frac{1}{4}\left[2 - 2\,\mathrm{Tr}\{\Lambda_0(\rho_0 - \rho_1)\}\right]$$

Bob has freedom in choosing POVM.
Thus,

$$p_e^* = \min_{\{\Lambda_0, \Lambda_1\}} p_e$$
$$= \frac{1}{4}\left[2 - \max_{\{\Lambda_0, \Lambda_1\}} 2\,\mathrm{Tr}\{\Lambda_0(\rho_0 - \rho_1)\}\right]$$
$$= \frac{1}{4}\left[2 - \|\rho_0 - \rho_1\|_1\right]$$

if $\rho_0 = \rho_1 \Rightarrow p_e^* = 1/2$ (just as good as a random guess)

if $\|\rho_0 - \rho_1\|_1 = 2 \Rightarrow p_e^* = 0$ perfectly distinguishable
(↑ maximum)

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Lemmas

Triangle Inequality (helpful for bounding errors)

$$\|\rho-\sigma\|_1 \le \|\rho-\tau\|_1 + \|\tau-\sigma\|_1$$

Proof:

$$\|\rho-\sigma\|_1 = 2\,\mathrm{Tr}\{\Pi^*(\rho-\sigma)\}$$
$$= 2\,\mathrm{Tr}\{\Pi^*(\rho-\tau)\} + 2\,\mathrm{Tr}\{\Pi^*(\tau-\sigma)\}$$
$$\le \|\rho-\tau\|_1 + \|\tau-\sigma\|_1$$

↑ b/c Π^* maybe not optimal for distinguishing ρ from τ & τ from σ

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Measurement on approximately close states

suppose Π : $0 \leq \Pi \leq I$

Then $\mathrm{Tr}\{\Pi\rho\} \geq \mathrm{Tr}\{\Pi\sigma\} - \|\rho-\sigma\|_1$

~~Most common use:~~

~~measurement~~ successful on σ : $\mathrm{Tr}\{\Pi\sigma\} \geq 1-\epsilon$

ρ close to σ : $\frac{1}{2}\|\rho-\sigma\|_1 \leq \epsilon$

then measurement should be successful on ρ :

$$\mathrm{Tr}\{\Pi\rho\} \geq 1-2\epsilon$$

Proof: $\frac{1}{2}\|\rho-\sigma\|_1 = \max_{0 \leq \Lambda \leq I} \mathrm{Tr}\{\Lambda(\sigma-\rho)\}$

$$\geq \mathrm{Tr}\{\Pi(\sigma-\rho)\}$$

$$= \mathrm{Tr}\{\Pi\sigma\} - \mathrm{Tr}\{\Pi\rho\}$$

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Monotonicity

states become less distinguishable after discarding:

$$\|\rho^A - \sigma^A\|_1 \le \|\rho^{AB} - \sigma^{AB}\|_1$$

Proof:

Suppose $\|\rho^A - \sigma^A\|_1 = 2\,\mathrm{Tr}\{\Lambda^A(\rho^A - \sigma^A$ for some Λ^A

then

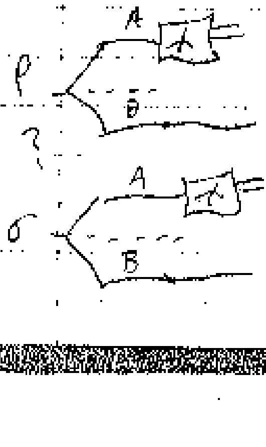
$$\|\rho^A - \sigma^A\|_1 = 2\,\mathrm{Tr}\{\Lambda^A(\rho^A - \sigma^A)\}$$

$$= 2\,\mathrm{Tr}\{(\Lambda^A \otimes I^B)(\rho^{AB} - \sigma^{AB})\}$$

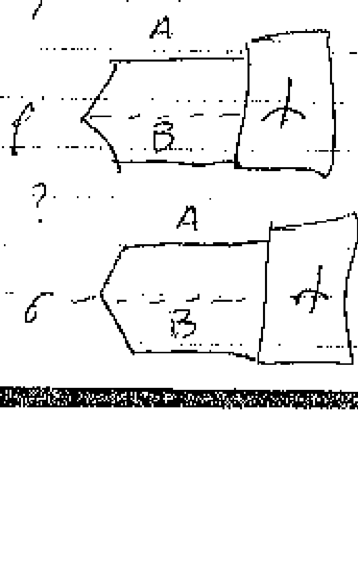
$$\le 2 \max_{0 \le \Lambda^{AB} \le I} \mathrm{Tr}\{\Lambda^{AB}(\rho^{AB} - \sigma^{AB})\}$$

$$= \|\rho^{AB} - \sigma^{AB}\|_1$$

Think hypothesis testing:



is worse than



Lecture 13

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Fidelity of two states

Begin w/ pure-state fidelity.

Suppose we input $|\psi\rangle$ into a protocol + it instead outputs $|\phi\rangle$:

Ideal: $|\psi\rangle^A \longrightarrow \boxed{I^{A\to B}} \longrightarrow |\psi\rangle^B$

Actual: $|\psi\rangle^A \longrightarrow \boxed{\text{Protocol}} \longrightarrow |\phi\rangle^B$

The pure-state fidelity is equal to the probability that $|\phi\rangle$ would pass a test for being $|\psi\rangle$:

$$F(|\psi\rangle, |\phi\rangle) \equiv |\langle\psi|\phi\rangle|^2$$

Thus, $0 \leq F(|\psi\rangle, |\phi\rangle) \leq 1$

0 if states are orthogonal +
1 if they are the same

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Most general form of fidelity for two mixed states ρ^A + σ^A — borrow idea of pure-state fidelity as overlap of pure states, but instead take purifications $|\phi_\rho\rangle^{RA}$ + $|\phi_\sigma\rangle^{RA}$

$$F(\rho,\sigma) \equiv \max_{|\phi_\rho\rangle^{RA}, |\phi_\sigma\rangle^{RA}} |\langle \phi_\rho | \phi_\sigma \rangle|^2$$

all purifications are the same up to unitaries on the reference

$$\therefore F(\rho,\sigma) = \max_{U_\rho^R, U_\sigma^R} |\langle \phi|_\rho \left((U_\rho^\dagger)^R \otimes I^A \right) (U_\sigma^R \otimes I^A) |\phi_\sigma\rangle|^2$$

$$= \max_{U_\rho^R, U_\sigma^R} |\langle \phi_\rho | (U_\rho^\dagger U_\sigma)^R \otimes I^A |\phi_\sigma\rangle|^2$$

$U_\rho^\dagger U_\sigma$ is just a single unitary, so

$$\therefore F(\rho,\sigma) = \max_{U^R} |\langle \phi_\rho | U^R \otimes I^A | \phi_\sigma \rangle|^2$$

Uhlmann fidelity

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Uhlmann's Theorem

$$F(\rho,\sigma) = \max_U |\langle\phi_\rho| U^R \otimes I^A |\phi_\sigma\rangle^{RA}|^2$$
$$= \|\sqrt{\rho}\sqrt{\sigma}\|_1^2$$

Recall: $\|A\|_1 = Tr\{\sqrt{A^\dagger A}\}$

$\therefore \|\sqrt{\rho}\sqrt{\sigma}\|_1^2$
$= Tr\{\sqrt{\sqrt{\rho}\sqrt{\sigma}\sqrt{\sigma}\sqrt{\rho}}\}^2$
$= Tr\{\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\}^2$

Proof: Suppose ρ has spectral decomposition

$$\rho = \sum_x p(x) |x\rangle\langle x|$$

can check that a purification of ρ is

$$|\phi_\rho\rangle \equiv \sqrt{d}\,(I^R \otimes \sqrt{\rho}^X)\,|\Phi^+\rangle^{RX}$$

where $|\Phi^+\rangle^{RX} \equiv \frac{1}{\sqrt{d}} \sum_i |i\rangle^R |i\rangle^X$

Similarly,

$$|\phi_\sigma\rangle \equiv \sqrt{d}\,(I^R \otimes \sqrt{\sigma}^X)\,|\Phi^+\rangle^{RX}$$

(holds regardless of basis of σ)

Let U^{*R} be the maximizing unitary

then
$$F(\rho,\sigma) = |\langle\phi_\rho| (U^*)^R \otimes I |\phi_\sigma\rangle|^2$$

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$$= \left| d \langle \Phi^+|^{RX} (U^*)^R \otimes \sqrt{\rho}^X \sqrt{\sigma}^X |\Phi^+\rangle^{RX} \right|^2$$

$$= \left| \sum_i \langle i|^R \langle i|^X (U^*)^R \otimes (\sqrt{\rho}\sqrt{\sigma})^X \sum_j |j\rangle^R |j\rangle^X \right|^2$$

$$= \left| \sum_{i,j} \langle i|^R \langle i|^X I^R \otimes (\sqrt{\rho}\sqrt{\sigma}(U^*)^T)^X |j\rangle^R |j\rangle^X \right|$$

$$= \left| \sum_{i,j} \langle i|j\rangle^R \langle i|^X (\sqrt{\rho}\sqrt{\sigma}(U^*)^T)^X |j\rangle^X \right|^2$$

$$= \left| \sum_i \langle i|^X \sqrt{\rho}\sqrt{\sigma}(U^*)^T |i\rangle^X \right|^2$$

$$= \left| \operatorname{Tr}\{ \sqrt{\rho}\sqrt{\sigma}(U^*)^T \} \right|^2 \quad (*)$$

Aside: Every operator A has a "right polar decomposition"

$$A = \sqrt{AA^\dagger}\, V = |A|\, V$$ (Analogous to $z = re^{i\theta}$)

Proof: Follows from SVD

Suppose $A = U_1 D U_2$

$\therefore AA^\dagger = U_1 D U_2 U_2^\dagger D U_1^\dagger$

$= U_1 D^2 U_1^\dagger$

$\therefore \sqrt{AA^\dagger} = U_1 D U_1^\dagger$

can take V as $U_1 U_2$

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Lemma: $|\mathrm{Tr}\{AU\}| \leq \mathrm{Tr}\{|A|\}$

w/ saturation when $U = V^\dagger$ where $A = |A|V$

Proof: $|\mathrm{Tr}\{AU\}| = |\mathrm{Tr}\{|A|VU\}|$

$= |\mathrm{Tr}\{|A|^{1/2}|A|^{1/2}VU\}|$ ← This is trace inner product $\mathrm{Tr}\{X^\dagger Y\}$ + we have $\mathrm{Tr}\{X^\dagger Y\} \leq \sqrt{\mathrm{Tr}\{X^\dagger X\}\,\mathrm{Tr}\{Y^\dagger Y\}}$

$\leq \sqrt{\mathrm{Tr}\{|A|\}\,\mathrm{tr}\{U^\dagger V^\dagger |A| U V\}}$

$= \mathrm{Tr}\{|A|\}$

going back, (back to (*))

$|\mathrm{Tr}\{\sqrt{\rho}\sqrt{\sigma}\,(U^*)^T\}|^2$

$\leq \mathrm{Tr}\{|\sqrt{\rho}\sqrt{\sigma}|\}^2$

$= \|\sqrt{\rho}\sqrt{\sigma}\|_1^2$

maximizing unitary is from right polar decomposition of $\sqrt{\rho}\sqrt{\sigma}$ □

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Properties of fidelity

Symmetry: $F(\rho, \sigma) = F(\sigma, \rho)$

evident from $\|\sqrt{\rho}\sqrt{\sigma}\|_1^2$ or

$$\max_{U^R} |\langle \phi_\rho | U^R | \phi_\sigma \rangle|^2$$

Monotonicity:

$$F(\rho^{AB}, \sigma^{AB}) \leq F(\rho^A, \sigma^A)$$

higher fidelity $\Leftrightarrow$ less distinguishable

Let $|\psi\rangle^{RAB}$ be a purification of ρ^{AB} & ρ^A

Let $|\phi\rangle^{RAB}$ be a purification of σ^{AB} & σ^A

Then $F(\rho^{AB}, \sigma^{AB}) = \max_{U^R \otimes I^{AB}} |\langle \psi|^{RAB} (U^R \otimes I^{AB}) |\phi\rangle^{RAB}|^2$

$F(\rho^A, \sigma^A) = \max_{U^{RB} \otimes I^A} |\langle \psi|^{RAB} (U^{RB} \otimes I^A) |\phi\rangle^{RAB}|$

this maximization is inclusive of the prior one

thus, $F(\rho^A, \sigma^A) \geq F(\rho^{AB}, \sigma^{AB})$

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Relationship between Trace Distance & Fidelity

$$1 - \sqrt{F(\rho,\sigma)} \le \frac{1}{2} \|\rho - \sigma\|_1 \le \sqrt{1 - F(\rho,\sigma)}$$

Suppose $F(\rho,\sigma) \ge 1 - \epsilon$ (two states are very similar)

Then $\epsilon \ge 1 - F(\rho,\sigma)$

$\therefore$ $\sqrt{\epsilon} \ge \sqrt{1 - F(\rho,\sigma)}$

$\therefore$ $\|\rho - \sigma\|_1 \le 2\sqrt{\epsilon}$ (trace distance should be small)

Similarly, suppose $\|\rho - \sigma\|_1 \le \epsilon$

$\therefore$ $1 - \sqrt{F(\rho,\sigma)} \le 1/2\, \epsilon$

$\therefore$ $\sqrt{F(\rho,\sigma)} \ge 1 - 1/2\, \epsilon$

$\therefore$ $F(\rho,\sigma) \ge (1 - 1/2\, \epsilon)^2$

$$= 1 - \epsilon + \frac{1}{4}\epsilon^2$$

$$\ge 1 - \epsilon$$

Let's prove $\frac{1}{2}\|\rho-\sigma\|_1 \leq \sqrt{1-F(\rho,\sigma)}$

Consider ρ + σ pure

Suppose $|\psi\rangle$ + $|\phi\rangle$ where

$$|\phi\rangle = \cos(\theta)\,|\psi\rangle + e^{i\varphi}\sin\theta\,|\psi^\perp\rangle$$

where $|\psi^\perp\rangle = \underbrace{(I - |\psi\rangle\langle\psi|)\,|\phi\rangle}_{\text{normalization}}$

Fidelity is then

$$|\langle\phi|\psi\rangle|^2 = \cos^2\theta$$

consider $|\phi\rangle\langle\phi|$

$$|\phi\rangle\langle\phi| = \begin{bmatrix} \cos^2\theta & e^{-i\varphi}\sin\theta\cos\theta \\ e^{i\varphi}\sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$ in basis $\{|\psi\rangle, |\psi^\perp\rangle\}$

$\therefore$ $|\psi\rangle\langle\psi| - |\phi\rangle\langle\phi|$

$$= \begin{bmatrix} 1-\cos^2\theta & -e^{-i\varphi}\sin\theta\cos\theta \\ e^{i\varphi}\sin\theta\cos\theta & -\sin^2\theta \end{bmatrix}$$

eigenvalues are $|\sin\theta|$ + $-|\sin\theta|$

$\therefore$ $\| \, |\psi\rangle\langle\psi| - |\phi\rangle\langle\phi| \, \|_1 = 2|\sin\theta|$

consider that

$$\left(\frac{2|\sin\theta|}{2}\right)^2 = 1 - \cos^2\theta$$

$$\therefore \left(\frac{\| |\psi\rangle\langle\psi| - |\phi\rangle\langle\phi| \|_1}{2}\right) = \sqrt{1 - F(|\psi\rangle, |\phi\rangle)}$$

To prove bound for mixed states ρ & σ, choose purifications $|\phi_\rho\rangle$ & $|\phi_\sigma\rangle$ such that

$$F(\rho^A, \sigma^A) = |\langle\phi_\sigma|\phi_\rho\rangle|^2 = F(|\phi_\rho\rangle^{RA}, |\phi_\sigma\rangle^{RA})$$

Then

$$\frac{1}{2}\|\rho^A - \sigma^A\|_1 \leq \frac{1}{2}\|\phi_\rho^{RA} - \phi_\sigma^{RA}\|_1$$

$$= \sqrt{1 - F(|\phi_\rho\rangle^{RA}, |\phi_\sigma\rangle^{RA})}$$

$$= \sqrt{1 - F(\rho^A, \sigma^A)}$$

Application: Gentle Measurement

(form of information-disturbance trade-off)

Lemma: Consider ρ & Λ such that $0 \leq \Lambda \leq I$.

Suppose $\text{Tr}\{\Lambda\rho\} \geq 1-\epsilon$ (*)
where $1 \geq \epsilon > 0$.

post-measurement state is

$$\rho' \equiv \frac{\sqrt{\Lambda}\,\rho\,\sqrt{\Lambda}}{\text{Tr}\{\Lambda\rho\}}.$$

(*) implies that measurement barely changes the state, in the sense that

$$\|\rho - \rho'\|_1 \leq 2\sqrt{\epsilon}$$

Proof: First suppose ρ is pure $|\psi\rangle\langle\psi|$

Post-measurement state is

$$\frac{\sqrt{\Lambda}|\psi\rangle\langle\psi|\sqrt{\Lambda}}{\langle\psi|\Lambda|\psi\rangle}$$

Fidelity between this state and original is

$$\langle\psi|\left(\frac{\sqrt{\Lambda}|\psi\rangle\langle\psi|\sqrt{\Lambda}}{\langle\psi|\Lambda|\psi\rangle}\right)|\psi\rangle$$

$$= \frac{|\langle\psi|\sqrt{\Lambda}|\psi\rangle|^2}{\langle\psi|\Lambda|\psi\rangle}$$

$$\geq \frac{|\langle\psi|\Lambda|\psi\rangle|^2}{\langle\psi|\Lambda|\psi\rangle} \qquad \left(\sqrt{\Lambda} \geq \Lambda \text{ when } \Lambda \leq I\right)$$

$$= \langle\psi|\Lambda|\psi\rangle$$

$$= \mathrm{Tr}\{\Lambda\,|\psi\rangle\langle\psi|\} \geq 1-\epsilon$$

Now consider mixed states

ρ^A ~~+~~ ρ'^A Let $|\psi\rangle^{RA}$ + $|\psi'\rangle^{RA}$ be purifications

$$|\psi'\rangle^{RA} = \frac{I^R \otimes \sqrt{\Lambda}^A |\psi\rangle^{RA}}{\langle\psi|^{RA} I^R \otimes \Lambda^A |\psi\rangle^{RA}}$$

$$\therefore \quad F(\rho^A, \rho'^A) \geq F(|\psi\rangle^{RA}, |\psi'\rangle^{RA}) \quad \text{(monotonicity)}$$

$$\geq 1-\epsilon$$

$\|\rho^A - \rho'^A\| \leq 2\sqrt{\epsilon}$ by relationship between trace distance + fidelity □

there are other useful variations of this lemma