Lecture 5

Schmidt decomposition

Given any bipartite vector $|\psi\rangle_{AB}$, we can always write it as

$$|\psi\rangle_{AB} = \sum_x f_x |\psi_x\rangle_A \otimes |\psi_x\rangle_B$$

for $\{|\psi_x\rangle_A\}$ & $\{|\psi_x\rangle_B\}$ O.N. bases.

Pf. Can expand $|\psi\rangle_{AB}$ in terms of O.N. bases $\{|l\rangle_A\}$ $\{|m\rangle_B\}$ as

$$|\psi\rangle_{AB} = \sum_{l,m} \alpha_{l,m} |l\rangle_A |m\rangle_B$$

But then let $\alpha_{l,m}$ be matrix elements of A. For any A we have SVD

$$A = UDV$$

↑ number of ≥ 0 entries here is $\min(d_A, d_B)$

so that

$$\alpha_{l,m} = \sum_{p} u_{l,p}\, d_{p,p}\, v_{p,m}$$

Substitute in to get

$$|\psi\rangle_{AB} = \sum_{l,m} \left(\sum_{p} u_{l,p}\, d_{p,p}\, v_{p,m} \right) |l\rangle_A \otimes |m\rangle_B$$

$$= \sum_{p} d_{p,p} \left(\sum_{l} u_{l,p} |l\rangle_A \right) \otimes \left(\sum_{m} v_{p,m} |m\rangle_B \right)$$

$$= \sum_{p} d_{p,p}\, |\phi_p\rangle_A \otimes |\psi_p\rangle_B$$

If we take

$$|\Gamma\rangle_{RA} = \sum_{i=1}^{d} |i\rangle_R \otimes |i\rangle_A$$

then

$$|\psi\rangle_{RB} = I_R \otimes (V)_{A\to B} |\Gamma\rangle_{RA}$$

for some $(V)_{A\to B}$

Pf.

let $|\psi\rangle_{RB} = \sum_{l,m} \alpha_{l,m} |l\rangle_R \otimes |m\rangle_B$

let $V_{A\to B} = \sum_{l,m} \alpha_{l,m} |m\rangle_B \langle l^*|_A$

each $|l\rangle_R = \sum_i |i\rangle\langle i|l\rangle$

So then

$(I_R \otimes V_{A\to B}) |\Gamma\rangle_{RA}$

$$= \sum_{l,m} \alpha_{l,m} |m\rangle_B \langle l^*|_A \sum_i |i\rangle_R \otimes |i\rangle_A$$

$$= \sum_{l,m,i} \alpha_{l,m} |i\rangle_R \otimes |m\rangle_B \langle l^*|i\rangle_A$$

$$= \sum_{l,m,i} \alpha_{l,m} \langle l^*|i\rangle_A |i\rangle_R \otimes |m\rangle_B$$

$$= \sum_{l,m,i} \alpha_{l,m} |i\rangle\langle i|_R |l\rangle_R \otimes |m\rangle_B$$

$$= \sum_{l,m} \alpha_{l,m} |l\rangle_R \otimes |m\rangle_B = |\psi\rangle_{RB}$$

then observe that

$$\langle i|_R \, |\psi\rangle_{RB} =$$

$$\langle i|_R \left(I_R \otimes V_{A\to B} \right) |\Gamma\rangle_{RA}$$

$$= V_{A\to B} \, \langle i|_R \, |\Gamma\rangle_{RA}$$

$$= V_{A\to B} \, \langle i|_R \sum_i |i\rangle_R \otimes |i\rangle_A$$

$$= V_{A\to B} \, |i\rangle_A$$

3 axioms for channels

Why linearity? We want

$$\mathcal{N}(\lambda\rho + (1-\lambda)\sigma) =$$

$$\lambda \mathcal{N}(\rho) + (1-\lambda)\mathcal{N}(\sigma)$$

perform any measurement of the measurement outcomes probabilities are given by

$$\mathrm{Tr}\{\Lambda_j (\mathcal{N}(\lambda\rho + (1-\lambda)\sigma))\} = p_J(j)$$

This should be equal to

$$\lambda\, p_J(j|\rho) + (1-\lambda)\, p_J(j|\sigma)$$

Could perform this experiment many times & the preparer of the state could later reveal which state was prepared. You could then check the consistency of the data. If you found a significant deviation, this would be a violation of linearity but we never observe this.

Any CPTP linear map can be written as

$\mathcal{N}(X) = \sum_{l} V_l X V_l^{\dagger}$ w/ $\sum_{l} V_l^{\dagger} V_l = I$

Proof of the Choi-Kraus theorem

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Abstract

We provide a simple proof of the Choi-Kraus theorem (as a reference).

Every evolution of a quantum state should satisfy three properties:

1. It should be linear so that we do not allow for signaling (by the steering argument, one could have signaling).
2. It should be completely positive so that it takes quantum states to quantum states (even for systems correlated with the one on which the map is acting).
3. It should be trace preserving (again so that it takes quantum states to quantum states).

The three requirements above lead naturally to Choi's theorem, which states that the map has to take a particular form according to a Kraus decomposition. We now give a very simple proof of Choi's theorem:

Let $|\Gamma\rangle_{RA}$ denote the following vector:

$$|\Gamma\rangle_{RA} = \sum_{i=1}^{d_A} |i\rangle_R \otimes |i\rangle_A , \tag{1}$$

so that it is equal to the maximally entangled state $|\Phi\rangle_{RA}$ times $\sqrt{d_A}$ since

$$|\Phi\rangle_{RA} = \frac{1}{\sqrt{d_A}} \sum_{i=1}^{d_A} |i\rangle_R \otimes |i\rangle_A . \tag{2}$$

Then consider the following Choi matrix of a completely-positive, trace-preserving (CPTP) linear map $\mathcal{N}_{A\to B}$:

$$\mathcal{N}_{A\to B}\left(|\Gamma\rangle\langle\Gamma|_{RA}\right) = \sum_{i,j=1}^{d_A} |i\rangle\langle j|_R \otimes \mathcal{N}_{A\to B}\left(|i\rangle\langle j|\right) . \tag{3}$$

This matrix completely describes the action of the map because it describes the action of it on every operator $|i\rangle\langle j|$, from which we can construct any other operator on which the map acts (it is a large

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$d_A d_B \times d_A d_B$ matrix with blocks of the form $\mathcal{N}_{A\to B}(|i\rangle\langle j|)$. Also, the above matrix is positive due to the requirement that the map is completely positive. So we can diagonalize $\mathcal{N}_{A\to B}(|\Gamma\rangle\langle\Gamma|_{RA})$ as follows:

$$\mathcal{N}_{A\to B}(|\Gamma\rangle\langle\Gamma|_{RA}) = \sum_{l=1}^{d_A d_B} |\phi_l\rangle\langle\phi_l|_{RB}. \tag{4}$$

(This decomposition does not necessarily have to be such that the vectors $\{|\phi_l\rangle_{RB}\}$ are orthonormal.) Consider by inspecting (3) that

$$(\langle i|_R \otimes I_B)(\mathcal{N}_{A\to B}(|\Gamma\rangle\langle\Gamma|_{RA}))(|j\rangle_R \otimes I_B) = \mathcal{N}_{A\to B}(|i\rangle\langle j|). \tag{5}$$

Now, we can define some operators $\{V_l\}$ by their action on a basis $\{|i\rangle_A\}$ as follows:

$$V_l |i\rangle_A = (\langle i|_R \otimes I_B)|\phi_l\rangle_{RB}. \tag{6}$$

This is related to the observation that any bipartite vector $|\phi_l\rangle_{RB}$ can be written as follows:

$$|\phi_l\rangle_{RB} = I_R \otimes (V_l)_B |\Gamma\rangle_{RA}, \tag{7}$$

for some operator V_l. After making this observation, we finally realize that it is possible to write

$$\mathcal{N}_{A\to B}(|i\rangle\langle j|) = (\langle i|_R \otimes I_B)(\mathcal{N}_{A\to B}(|\Gamma\rangle\langle\Gamma|_{RA}))(|j\rangle_R \otimes I_B) \tag{8}$$

$$= (\langle i|_R \otimes I_B)\sum_{l=1}^{d_A d_B} |\phi_l\rangle\langle\phi_l|_{RB}(|j\rangle_R \otimes I_B) \tag{9}$$

$$= \sum_{l=1}^{d_A d_B} [(\langle i|_R \otimes I_B)|\phi_l\rangle_{RB}][\langle\phi_l|_{RB}(|j\rangle_R \otimes I_B)] \tag{10}$$

$$= \sum_{l=1}^{d_A d_B} V_l |i\rangle\langle j|_A V_l^\dagger. \tag{11}$$

By linearity of the map $\mathcal{N}_{A\to B}$, it follows that its action on any operator σ can be written as follows:

$$\mathcal{N}_{A\to B}(\sigma) = \sum_{l=1}^{d_A d_B} V_l \sigma V_l^\dagger, \tag{12}$$

since any operator σ can be written as a linear combination of operators in the basis $\{|i\rangle\langle j|\}$.

If the decomposition in (4) is the spectral decomposition, then it follows that the Kraus operators $\{V_l\}$ are orthogonal with respect to the Hilbert-Schmidt inner product:

$$\mathrm{Tr}\left\{V_l^\dagger V_k\right\} = \mathrm{Tr}\left\{V_l^\dagger V_l\right\}\delta_{l,k}. \tag{13}$$

This follows from the fact that

$$\delta_{l,k}\langle\phi_l|\phi_l\rangle = \langle\phi_l|\phi_k\rangle \tag{14}$$
$$= \langle\Gamma|_{RB}\left[I_R\otimes\left(V_l^\dagger V_k\right)_B\right]|\Gamma\rangle_{RB} \tag{15}$$
$$= \sum_{i,j}\langle i|j\rangle\ \langle i|\left(V_l^\dagger V_k\right)|j\rangle \tag{16}$$
$$= \sum_{i}\langle i|\left(V_l^\dagger V_k\right)|i\rangle \tag{17}$$
$$= \text{Tr}\left\{V_l^\dagger V_k\right\}. \tag{18}$$

The Kraus operators $\{V_l\}$ should satisfy the following condition:

$$\sum_{l=1}^{d_A d_B} V_l^\dagger V_l = I. \tag{19}$$

This follows from the fact that

$$\text{Tr}_B\left\{\mathcal{N}_{A\to B}\left(|\Phi\rangle\langle\Phi|_{RA}\right)\right\} = \frac{1}{d_A}I_R. \tag{20}$$

(That is, when inputting the maximally entangled state, the reduced state on the reference should be the maximally mixed state no matter what the CPTP map $\mathcal{N}_{A\to B}$ is.) By using the fact that

$$(I\otimes C_A)|\Phi\rangle_{RA} = \left(C_R^T\otimes I_A\right)|\Phi\rangle_{RA}, \tag{21}$$

for any operator C, we find that

$$\text{Tr}_B\left\{\mathcal{N}_{A\to B}\left(|\Phi\rangle\langle\Phi|_{RA}\right)\right\} = \text{Tr}_B\left\{\sum_{l=1}^{d_A d_B}(I_R\otimes V_l)\left(|\Phi\rangle\langle\Phi|_{RA}\right)\left(I_R\otimes V_l^\dagger\right)\right\} \tag{22}$$
$$= \text{Tr}_B\left\{\sum_{l=1}^{d_A d_B}\left(\left(V_l^T\right)_R\otimes I_A\right)\left(|\Phi\rangle\langle\Phi|_{RA}\right)\left(\left(V_l^*\right)_R\otimes I_A\right)\right\} \tag{23}$$
$$= \frac{1}{d_A}\sum_{l=1}^{d_A d_B} V_l^T V_l^* \tag{24}$$
$$= \frac{1}{d_A}I_R. \tag{25}$$

The result that $\sum_{l=1}^{d_A d_B} V_l^\dagger V_l$ follows because

$$\frac{1}{d_A}I_R = \left(\frac{1}{d_A}I_R\right)^*. \tag{26}$$

Trace preserving

$$\mathrm{Tr}\{X\} = \mathrm{Tr}\{\mathcal{N}(X)\}$$

$$\mathcal{N}(X) = \sum_l V_l X V_l^\dagger$$

$$\mathrm{Tr}\{\mathcal{N}(X)\} = \mathrm{Tr}\{(\sum_l V_l^\dagger V_l) X\}$$

If $\sum_l V_l^\dagger V_l = I$

then traces preserving

$$\delta_{ij} = \mathrm{Tr}\{|i\rangle\langle j|\} = \mathrm{Tr}\{\mathcal{N}(|i\rangle\langle j|)\}$$

$$= \mathrm{Tr}\{\sum_l V_l |i\rangle\langle j| V_l^\dagger\}$$

$$= \langle i| \left(\sum_l V_l^\dagger V_l\right) |j\rangle$$

$$= \delta_{ij}$$

holds for all bases

$\Rightarrow$

$$\sum_l V_l^\dagger V_l = I$$

$\sum_i |i\rangle\langle i|l\rangle =$

~~take~~

let $|\Psi\rangle_{RB} = \sum_{l,m} \alpha_{l,m} |l\rangle_R \otimes |m\rangle_B$

take $V =$ ~~$\sum_{l,m} \alpha_{l,m} |B\rangle$~~

let ~~$|\Psi\rangle_{RB} = \sum_p d_p |\phi_p\rangle\langle\phi_p|_R$~~

$$V_{A\to B} = \sum_{l,m} \alpha_{l,m} |m\rangle_B \langle l^*|_A$$

$$\sum_{l,m} \alpha_{l,m} |m\rangle_B \langle l|_A \sum_i |i\rangle_R \otimes |i$$

$$= \sum_{l,m,i} \alpha_{l,m} |i\rangle_R \otimes |m\rangle_B \langle l^*|i\rangle$$

$$= \sum_{l,m,i} \alpha_{l,m} |i\rangle_R \langle i|l\rangle \otimes |m\rangle_B$$

$$= \sum_{l,m} \alpha_{l,m} |l\rangle_R \otimes |m\rangle_B$$