To appear in the Proceedings of the 8th International Conference on QCMC

QUANTUM FORBIDDEN-INTERVAL THEOREMS FOR STOCHASTIC RESONANCE WITH SQUEEZED LIGHT

Mark M. Wilde and Bart Kosko

Signal and Image Processing Institute, Department of Electrical Engineering, University of Southern California, Los Angeles, California, USA 90089

We show that quantum optics extends the classical "forbidden interval" theorem that produces a stochastic-resonance noise benefit in threshold systems when detecting subthreshold signals. The new quantum forbidden interval theorems show that a noise benefit occurs if and only if the position quadrature of the noise mean does not lie in a given interval. The result holds for all types of finite-variance noise and for all infinite-variance stable noise. These guaranteed noise benefits suggest loopholes in the security of continuous-variable quantum key distribution with subthreshold signals.

Noise can help some nonlinear systems process signals in the so-called stochastic resonance (SR) effect [1]. Figure 2 shows the nonmonotonic signature of an SR noise benefit in a quantum-optical communication system. The bit count rises and then falls as the intensity of the added noise increases. The SR effect does not occur in the linear communication systems of classical information theory where noise can only degrade the linear system's performance [2, 3]. The SR effect admits a simple mathematical characterization in the special case of memoryless threshold neurons or other threshold functions for threshold T and subthreshold signals A and $-A$ such that $-A < A < T$. The basic forbidden interval theorem states that the threshold system exhibits SR if and only if the noise mean $E(n)$ does not lie in the interval $(T - A, T + A)$ for any noise probability density function with finite variance [4, 5]. The same result holds for stable infinite variance noise [6] when the noise location parameter replaces the mean. We now show how to extend these SR theorems to quantum optics using squeezed light [7].

The quantum forbidden-interval theorems apply to the quantum-optical communication system of Figure 1. This system assumes weak or subthreshold signals and further assumes that noise corrupts these signals. Alice sends a squeezed displaced vacuum as a binary signal to Bob. Then Bob decodes the binary message by position-quadrature homodyning and thresholding.

We develop the quantum-optical model in the Heisenberg picture. We consider only the position-quadrature operator's evolution because Bob performs a position-quadrature homodyne detection at the end of the protocol. Suppose Alice possesses a vacuum mode. Let $\hat{x}$ denote the position-quadrature operator of this vacuum state. This operator collapses to a zero-mean 1/2-variance Gaussian random variable X if Alice measures her state. Suppose that Alice does not measure it. Suppose instead that she sends her mode through a squeezer. Suppose further that she can control the strength of squeezing with a squeezing parameter r. Her position-quadrature operator $\hat{x}$ evolves under the squeezer as

$$\hat{x} \xrightarrow{\text{squeezer}} \hat{x}e^{-r} \tag{1}$$

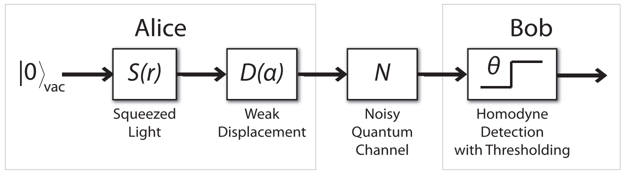


Figure 1: Noisy quantum-optical communication channel for stochastic resonance (SR).

She encodes a random message bit $S \in \{0,1\}$ by displacing her state by $\alpha \in \mathbb{C}$ if $S=1$ or by $-\alpha$ if $S=0$. Her operator evolves under the displacement as

$$\hat{x}e^{-r} + (-1)^{S+1}\alpha_x \tag{2}$$

where $\alpha_x = \mathrm{Re}\{\alpha\}$. She sends her state to Bob over an additive noisy bosonic channel [8]. A noisy bosonic channel affects any annihilation operator $\hat{a}_{in}$ by

$$\hat{a}_{out} = \hat{a}_{in} + \nu \tag{3}$$

Annihilation operator $\hat{a}_{out}$ represents the output mode. Complex random variable ν represents the noisy effects of the bosonic channel. Random variable ν need not be Gaussian for the SR effect to occur—it can have finite variance or possess an alpha-stable distribution [6]. Bob receives the state

$$\hat{x}e^{-r} + (-1)^{S+1}\alpha_x + \nu_x \tag{4}$$

from the noisy channel where $\nu_x = \mathrm{Re}\{\nu\}$. Bob performs a position-quadrature homodyne detection so that the state collapses to the random variable

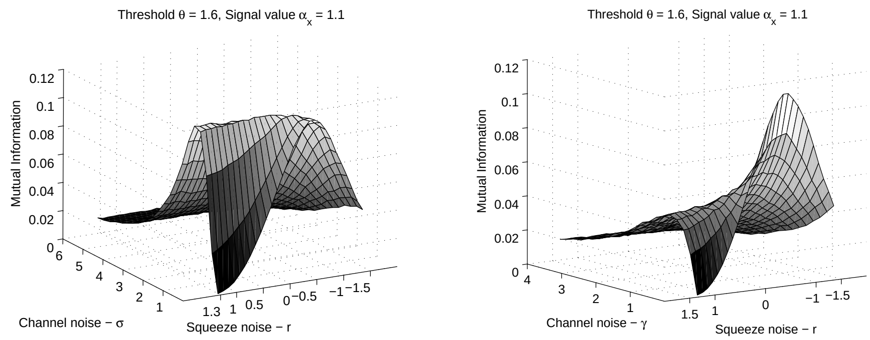
$$(-1)^{S+1}\alpha_x + N \tag{5}$$

where $N = Xe^{-r} + \nu_x$ sums both noisy random variables. Bob thresholds the result of this homodyne detection with threshold θ to retrieve a bit Y. This bit Y should be the message bit S that Alice first sent. Random variables Xe^{-r} and ν_x are independent because random variable Xe^{-r} originates from vacuum fluctuations and because ν_x represents Bob's loss of knowledge due to the state's propagation through a noisy quantum channel. The density $p_N(n)$ of random variable N is

$$p_N(n) = (p_{Xe^{-r}} * p_{\nu_x})(n) \tag{6}$$

where $p_{Xe^{-r}}(n)$ is the density of a zero-mean $e^{-2r}/2$-variance Gaussian random variable, $p_{\nu_x}(n)$ is the density of ν_x, and $*$ denotes convolution. The first quantum forbidden-interval theorem gives necessary and sufficient conditions for the nonmonotone SR effect.

Theorem 1 *Suppose the channel noise's position quadrature has finite variance σ_x^2 and mean μ_x. Suppose the input signal's position quadrature α_x is subthreshold: $\alpha_x < \theta$. Suppose there is some statistical dependence between input signal S and output signal Y so that the mutual information obeys $I(S,Y) > 0$. Then the quantum communication system exhibits the nonmonotone SR effect if and only if the position quadrature of the noise mean does not lie in the forbidden interval: $\mu_x \notin [\theta - \alpha_x, \theta + \alpha_x]$. The nonmonotone SR effect is that $I(S,Y) \to 0$ as $\sigma_x^2 \to 0$ and as squeezing parameter $r \to \infty$.*



(a) (b)

Figure 2: Nonmonotone SR noise benefits for (a) white Gaussian noise per Theorem 1 and (b) white Cauchy (infinite-variance) noise per Theorem 2.

Proof. The proof for sufficiency and necessity follows the proof method in [4] and [5] respectively if we use (6) as the noise density. ■

Theorem 2 applies to a quantum alpha-stable noise source. Stable models apply to diverse physical phenomena that include impulsive interrupts in phone lines, underwater acoustics, low-frequency atmospheric signals, and gravitational fluctuations [6]. Symmetric alpha-stable noise [6] results from an impulsive noise source and describes a family of thick-tailed bell-curve densities. The parameter α (which differs from the coherent state α) lies in $(0, 2]$ and governs the thickness of the distribution's tail: $\alpha = 1$ corresponds to the thick-tailed Cauchy random variable and $\alpha = 2$ corresponds to the familiar thin-tailed Gaussian random variable. The bell curve's tail thickness increases as α decreases. The generalized central limit theorem states that all and only normalized stable random variables converge in distribution to a stable random variable [9].

Theorem 2 *Suppose the position quadrature of the channel noise is a quantum alpha-stable noise source with dispersion γ_x and location a_x. Suppose the position quadrature α_x of the input signal is subthreshold such that $\alpha_x < \theta$. Suppose there is some statistical dependence between input signal S and output signal Y so that the mutual information obeys $I(S, Y) > 0$. Then the quantum communication system exhibits the nonmonotone SR effect if and only if the position quadrature of the noise location does not lie in the forbidden interval: $a_x \notin [\theta - \alpha_x, \theta + \alpha_x]$. The nonmonotone SR effect is that $I(S, Y) \to 0$ as $\gamma_x \to 0$ and as squeezing parameter $r \to \infty$.*

Proof. The proof for sufficiency and necessity follows the stable proof method in [4] and [5] respectively if we use (6) as the noise density and if ν_x in (4) is an alpha-stable random variable. ■

Figure 2 shows simulation instances of the two SR theorems. Figure 2 displays the full "inverted-U" curve for realistic squeezing values. A value of 1.35 for squeezing parameter r corresponds to 14.88 dB. Experimentalists have realized squeezing of 10 dB [10].

The subthreshold conditions for the SR effect can arise in continuous-variable quantum key distribution (CVQKD) with thresholding and subthreshold signals [11]. Then the SR effect gives a noise benefit to an attacker Eve. Eve leaks a small fraction of the signal through a beamsplitter and adds an optimal amount of noise. Eve gains knowledge about the secret key in this fashion by increasing her mutual information with the sender.

Theorems 1 and 2 guarantee only that the nonmonotone SR effect occurs. They do not give the optimal combination of channel noise and squeezing or guarantee a large increase in mutual information. Another criticism is that the subthreshold conditions need not occur in a quantum optical communication scenario because Bob can set the threshold level. Communication is not optimal if he sets the threshold so that the signals are subthreshold. The theorems apply only when the signals are subthreshold and only when the receiver does not have control over the threshold value. But this loss of threshold-level control occurs in the above CVQKD scenario. Alice and Bob agree on a high threshold level. Alice sends subthreshold signals but Eve has no control over the threshold level.

A further criticism is that the theorems are not realistic because they require infinite squeezing and thus require infinite energy to produce the SR effect. But the theorems guarantee that the SR effect occurs for some finite squeezing. The simulations in Figure 2 display the full joint σ^2 and r nonmonotone SR signature for experimentally plausible squeezing values and for realistic channel noise levels. The quantum forbidden-interval theorems are broad because they apply whenever Alice sends a subthreshold signal and Bob thresholds. So there should be other cases in quantum communication where SR noise benefits occur.

The authors thank Austin Lund for stimulating discussions. They thank Todd A. Brun, Igor Devetak, and Jonathan P. Dowling for helpful comments and thank the Hearne Institute for Theoretical Physics for financial support.

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