Unified Quantum Convolutional Coding

Mark M. Wilde and Todd A. Brun

Abstract—We outline a quantum convolutional coding technique for protecting a stream of classical bits and qubits. Our goal is to provide a framework for designing codes that approach the "grandfather" capacity of an entanglement-assisted quantum channel for sending classical and quantum information simultaneously. Our method incorporates several resources for quantum redundancy: fresh ancilla qubits, entangled bits, and gauge qubits. The use of these diverse resources gives our technique the benefits of both active and passive quantum error correction. We can encode a classical-quantum bit stream with periodic quantum gates because our codes possess a convolutional structure. We end with an example of a "grandfather" quantum convolutional code that protects one qubit and one classical bit per frame by encoding them with one fresh ancilla qubit, one entangled bit, and one gauge qubit per frame. We explicitly provide the encoding and decoding circuits for this example and discuss its error-correcting capability.

Index Terms—grandfather quantum convolutional codes, entanglement-assisted quantum convolutional codes

I. Introduction

The goal of quantum Shannon theory is to quantify the amount of quantum communication, classical communication, and entanglement required for various information processing tasks [1], [2], [3], [4]. Quantum teleportation and superdense coding [5] provided the initial impetus for quantum Shannon theory because these protocols demonstrate that we can combine entanglement, noiseless quantum communication, and noiseless classical communication to transmit quantum or classical information. In practice, the above resources are not noiseless because quantum systems decohere by interacting with their surrounding environment. Quantum Shannon theory is a collection of capacity theorems that determine the ultimate limits for noisy quantum communication channels. Essentially all quantum protocols have been unified as special cases of a handful of abstract protocols [4].

The techniques in quantum Shannon theory determine the asymptotic limits for communication, but these techniques do not produce practical ways of realizing these limits. This same practical problem exists with classical Shannon theory because the proofs that the channel capacities for classical communication are achievable use random coding techniques that are too inefficient in practice [6].

An example of an important capacity theorem from quantum Shannon theory results from the "father" protocol [4]. The father capacity theorem determines the optimal trade-off between the rate E of ebits (entangled qubits in the state $|\Phi^+\rangle^{AB} \equiv (|00\rangle^{AB} + |11\rangle^{AB})/\sqrt{2}$) and the rate Q of qubits in

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entanglement-assisted quantum communication.[1] These rates for quantum communication and entanglement consumption (or generation if E is negative) fall within a two-dimensional capacity region. Suppose that a noisy quantum channel $\mathcal{N}$ connects a sender to a receiver. Let $[q \rightarrow q]$ denote one qubit of noiseless quantum communication and let $[qq]$ denote one ebit of entanglement. The following resource inequality is a statement of the capability of the father protocol:

$$\langle \mathcal{N} \rangle + E[qq] \geq Q[q \rightarrow q]. \qquad (1)$$

The above resource inequality states that n uses of the noisy quantum channel $\mathcal{N}$ and nE noiseless ebits are sufficient to communicate nQ noiseless qubits in the limit of large n. The rates E and Q are related to the noisy channel $\mathcal{N}$ and there is a mutual dependence between them so that they form a capacity region. The father capacity theorem gives the optimal limits on the resources, but it does not provide a useful quantum coding technique for approaching the above limits.

Another important capacity theorem determines the ability of a noisy quantum channel to send "classical-quantum" states [7]. Let $[c \rightarrow c]$ denote one classical bit of noiseless classical communication. The result of the classical-quantum capacity theorem is also a resource inequality:

$$\langle \mathcal{N} \rangle \geq Q[q \rightarrow q] + R[c \rightarrow c]. \qquad (2)$$

The resource inequality states that n uses of the noisy quantum channel $\mathcal{N}$ are sufficient to communicate nQ noiseless qubits and nR noiseless classical bits in the limit of large n. The capacity theorem associated to the above resource inequality, in some cases, proves that we can devise clever classical-quantum codes that perform better than time-sharing a noisy quantum channel $\mathcal{N}$ between purely quantum codes and purely classical codes.

The "grandfather" capacity theorem determines the optimal triple trade-off between qubits, ebits, and classical bits for simultaneous transmission of classical and quantum information using an entanglement-assisted noisy quantum channel $\mathcal{N}$ [8]. The grandfather resource inequality is as follows:

$$\langle \mathcal{N} \rangle + E[qq] \geq Q[q \rightarrow q] + R[c \rightarrow c]. \qquad (3)$$

The above resource inequality is again an asymptotic statement and its meaning is similar to that in (1) and (2). The optimal rates in the above resource inequality coincide with the father inequality (1) when $R = 0$, with the classical-quantum inequality (2) when $E = 0$, and with the quantum capacity [1], [2], [3] when both $R = 0$ and $E = 0$. The

optimal strategy for the grandfather protocol is not time-sharing the channel between father codes and entanglement-assisted classical codes. It remains to be proven whether this optimal strategy outperforms time-sharing [8].

The goal of quantum error correction [5] is to find efficient and practical ways of coding quantum information to protect it against decoherence. One aspiration for this theory is to find quantum codes that approach the rates given by quantum Shannon theory in the limit of large block size.

The entanglement-assisted stabilizer formalism is a method for building a quantum block code using entanglement [9]. This theory has several benefits, such as the ability to produce a quantum code from an arbitrary classical linear block code, and has several generalizations [10], [11], [12]. Entanglement-assisted codes are "father" codes, in the sense that a good entanglement-assisted coding strategy should approach the optimal rates in (1).

In this Proceeding, we design a framework for "grandfather" quantum codes. Our grandfather codes are useful for the simultaneous transmission of classical and quantum information. Rather than using block codes for this purpose, we design quantum convolutional codes.[2] Quantum convolutional coding is a recent extension of the stabilizer formalism [14], [15], [16]. One of the benefits of a convolutional code is that it encodes a stream of information with an online periodic encoding circuit. Our technique incorporates many of the known techniques for quantum coding: subsystem codes [17], [18], entanglement-assisted codes [9], convolutional codes [14], [15], [16], and classical-quantum coding [7], [19], [13]. The goal of our technique is to provide a formalism for designing codes that approach the optimal triple trade-off rates in the grandfather resource inequality in (3). We are not claiming that codes in our framework will reach capacity, but we are instead providing a framework that one might later incorporate in a larger theory, such as a quantum turbo coding theory [21].

We structure this proceeding as follows. Section II details our "grandfather" quantum convolutional codes. We detail the finite-depth operations that quantum convolutional circuits employ when encoding and decoding a stream of quantum information. We explicitly show how to encode a stream of classical-quantum information using finite-depth operations and discuss the error-correcting properties of our codes. We end with an example of a grandfather quantum convolutional code. We discuss errors that the code corrects actively and others that it corrects passively.

II. GRANDFATHER QUANTUM CONVOLUTIONAL CODES

We now detail the stabilizer formalism for our grandfather quantum convolutional codes and describe how these codes operate. These codes are a significant extension of the existing entanglement-assisted quantum convolutional codes [12].

An $[n, k, l; r, c]$ grandfather quantum convolutional code encodes k information qubits and l information classical bits

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[2]Kremsky, Hsieh, and Brun address the formulation of grandfather block codes in a recent article [13].

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with the help of c ebits, $a = n - k - l - c - r$ ancilla qubits, and r gauge qubits. Each input frame includes the following:

1) The sender Alice's half of c ebits in the state $|\Phi^+\rangle$.
2) $a = n - k - c - l - r$ ancilla qubits in the state $|0\rangle$.
3) r gauge qubits (which can be in any arbitrary state σ).
4) l classical information bits $x^1 \cdots x^l$, given by a computational basis state $|x\rangle = X^{x^1} \otimes \cdots \otimes X^{x^l} |0\rangle^{\otimes l}$.
5) k information qubits in an arbitrary pure state $|\psi\rangle$.[3]

The left side of Figure 1 shows an example initial qubit stream before an encoding circuit operates on it.

The stabilizer matrix $S_0(D)$ for the initial qubit stream is as follows:

$$S_0(D) = \left[\begin{array}{cccccc|cccccc} I & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right],$$

(4)

where all identity matrices in the first two sets of rows are $c \times c$, the identity matrix in the last row is $a \times a$, the three columns of all zeros in both the "Z" and "X" matrices are respectively $(a + 2c) \times r$, $(a + 2c) \times l$, and $(a + 2c) \times k$. The matrices on the left of the vertical bar form the "Z" matrix and those on the right form the "X" matrix according to the Pauli-to-binary-polynomial isomorphism (see Ref.'s [10], [12] for a review of this isomorphism from the set of Pauli sequences to vectors of binary polynomials). The first two sets of rows stabilize a set of c ebits and the last set of rows stabilizes a set of a ancilla qubits. The first c columns of both the "Z" and "X" matrix correspond to halves of ebits that the receiver Bob possesses and the last n columns in both matrices correspond to the qubits that Alice possesses. The global generators (including both Alice and Bob's qubits) form a commuting set of generators for all shifts, but Alice's local generators do not necessarily form a commuting set.

Different sets of generators for the grandfather code are important in active error correction, in passive error correction, and for the identification of the l classical information bits. We first write the unencoded generators that act on the initial qubit stream. The first subgroup of generators is the entanglement subgroup $\mathcal{S}_{E,0}$ with the following generators:

$$S_{E,0}(D) = \left[\begin{array}{ccccc|ccccc} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \end{array} \right]. \quad (5)$$

The above generators are equivalent to the first two sets of rows in (4) acting on Alice's n qubits. The next subgroup is the isotropic subgroup $\mathcal{S}_{I,0}$ with the following generators:

$$S_{I,0}(D) = \left[\begin{array}{ccccc|ccccc} 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]. \quad (6)$$

The above generators are equivalent to the last set of rows in (4) acting on Alice's n qubits. The encoded versions of both of the above two matrices are important for the active correction of errors. The next subgroup is the gauge subgroup $\mathcal{S}_{G,0}$ whose generators are as follows:

$$S_{G,0}(D) = \left[\begin{array}{ccccc|ccccc} 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \end{array} \right]. \quad (7)$$

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[3]This statement is not entirely true because the information qubits can be entangled across multiple frames, or with an external system, but we use it to illustrate the idea.

The generators in $\mathcal{S}_{G,0}$ correspond to quantum operations that have no effect on the encoded quantum information and therefore represent a set of errors to which the code is immune. The last subgroup is the classical subgroup $\mathcal{S}_{C,0}$ with generators

$$S_{C,0}(D) = \begin{bmatrix} 0 & 0 & 0 & I & 0 & | & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (8)$$

The grandfather code passively corrects errors corresponding to the encoded version of the above generators because the initial qubit stream is immune to the action of operators in $\mathcal{S}_{C,0}$ (up to a global phase). Alice could measure the generators in $\mathcal{S}_{C,0}$ to determine the classical information in each frame. Unlike quantum information, it is possible to measure classical information without disturbing it.

Alice performs a periodic encoding circuit on her qubits to encode the initial set of ebits, ancilla qubits, and information qubits in each frame. She performs encoding operations only on her qubits because the channel spatially separates her qubits from Bob's qubits. The periodic encoding circuit encodes the information qubits and transforms the initial set of generators in (5-8) to a more general set of encoded generators. We use three types of operations in the example code in Section III:

1) Let $H(i)$ denote a Hadamard gate acting on qubit i of every frame. The effect of $H(i)$ is to swap column i in the "Z" matrix with column i in the "X" matrix.

2) Let $C(i, j, D^k)$ denote a CNOT gate from qubit i in every frame to qubit j in a frame delayed by k where $i \neq j$. This gate affects both the "X" and "Z" matrices. In the "X" matrix, it multiplies column i by D^k and adds the result to column j. In the "Z" matrix, it multiplies column j by D^{-k} and adds the result to column i. Let $f(D)$ be an arbitrary finite binary polynomial. Let $C(i, j, f(D))$ denote the sequence of CNOT gates corresponding to the polynomial $f(D)$.

3) Let $S(i, j)$ swap qubits i and j in every frame and column i and column j in both "X" and "Z."

Quantum convolutional circuits can employ other operations besides the above three [15], but we need only these three operations for the purposes of the current paper. The above operations and the others in the above references are *finite-depth*, because they transform any Pauli sequence with a finite number of non-identity entries to a Pauli sequence with a finite number of non-identity entries [12]. Finite-depth operations are desirable because they do not propagate uncorrected errors into the qubit stream when encoding or decoding.

The three gates used in this paper are all their own inverses. Therefore, the operations of the decoding circuit are the encoding operations performed in reverse order. The online nature of the decoding circuit follows directly from the online nature of the encoding circuit.

Alice performs an encoding circuit with finite-depth operations to encode her stream of qubits before sending them over the noisy quantum channel. The encoding circuit transforms the initial stabilizer $S_0(D)$ to the encoded stabilizer $S(D)$ as follows:

$$S(D) = \begin{bmatrix} I & Z_{E1}(D) & 0 & X_{E1}(D) \\ 0 & Z_{E2}(D) & I & X_{E2}(D) \\ 0 & Z_I(D) & 0 & X_I(D) \end{bmatrix}, \quad (9)$$

where $Z_{E1}(D)$, $X_{E1}(D)$, $Z_{E2}(D)$ and $X_{E2}(D)$ are each $c \times n$-dimensional, and $Z_I(D)$ and $X_I(D)$ are both $a \times n$-dimensional. The encoding circuit affects only the rightmost n entries in both the "Z" and "X" matrix of $S_0(D)$ because these are the qubits in Alice's possession. It transforms $S_{E,0}(D)$, $S_{I,0}(D)$, $S_{G,0}(D)$, and $S_{C,0}(D)$ as follows:

$$S_E(D) = \begin{bmatrix} Z_{E1}(D) & X_{E1}(D) \\ Z_{E2}(D) & X_{E2}(D) \end{bmatrix}, \quad (10)$$

$$S_I(D) = \begin{bmatrix} Z_I(D) & | & X_I(D) \end{bmatrix}, \quad (11)$$

$$S_G(D) = \begin{bmatrix} Z_{G1}(D) & X_{G1}(D) \\ Z_{G2}(D) & X_{G2}(D) \end{bmatrix}, \quad (12)$$

$$S_C(D) = \begin{bmatrix} Z_C(D) & | & X_C(D) \end{bmatrix}, \quad (13)$$

where $Z_{G1}(D)$, $X_{G1}(D)$, $Z_{G2}(D)$ and $X_{G2}(D)$ are each $r \times n$-dimensional and $Z_C(D)$ and $X_C(D)$ are each $l \times n$-dimensional. The above polynomial matrices have the same commutation relations as their corresponding unencoded polynomial matrices in (5-8) and respectively generate the entanglement subgroup $\mathcal{S}_E$, the isotropic subgroup $\mathcal{S}_I$, the gauge subgroup $\mathcal{S}_G$, and the classical subgroup $\mathcal{S}_C$.

The condition for a set of generators to form a commuting stabilizer is equivalent to orthogonality of each row in $S(D)$ with respect to the shifted symplectic product [14], [10]. This is equivalent to the condition

$$Z(D^{-1}) X^T(D) + X(D^{-1}) Z^T(D) = 0, \quad (14)$$

where $+$ represents binary addition of polynomials and the above matrix on the right of the equality is an $(n-k) \times (n-k)$ null matrix. The original generators in (4) obey this condition, and the periodic encoding circuit preserves the condition because any encoding circuit preserves the commutation relations of the original generators.

A grandfather quantum convolutional code operates as follows. Alice begins with an initial qubit stream as above. She performs the finite-depth encoding operations corresponding to a specific grandfather quantum convolutional code. She sends the encoded qubits online over the noisy quantum communication channel. Bob combines the received qubits with his half of the ebits in each frame. He obtains the error syndrome by measuring the generators in (9). He processes these syndrome bits with a classical error estimation algorithm to diagnose errors and applies recovery operations to reverse the errors. He then performs the inverse of the encoding circuit to recover the initial qubit stream with the information qubits and the classical information bits. He recovers the classical information bits either by measuring the generators in $\mathcal{S}_C$ before decoding or the generators in $\mathcal{S}_{C,0}$ after decoding.

A grandfather quantum convolutional code corrects errors in a Pauli error set $\mathcal{E}$ that obey one of the following conditions $\forall E_a, E_b \in \mathcal{E}$:

$$\exists \, g \in \langle \mathcal{S}_I, \mathcal{S}_E \rangle : \{g, E_a^\dagger E_b\} = 0 \ \ \text{or} \ \ E_a^\dagger E_b \in \langle \mathcal{S}_I, \mathcal{S}_G, \mathcal{S}_C \rangle,$$

where $\langle \cdot \rangle$ denotes the larger group generated by a set of subgroups and $\{A, B\}$ denotes the anticomutator for two operators A and B so that $\{A, B\} \equiv AB + BA$. It corrects errors that anticommute with generators in $\langle \mathcal{S}_I, \mathcal{S}_E \rangle$ by employing a classical error estimation algorithm, such as the Viterbi
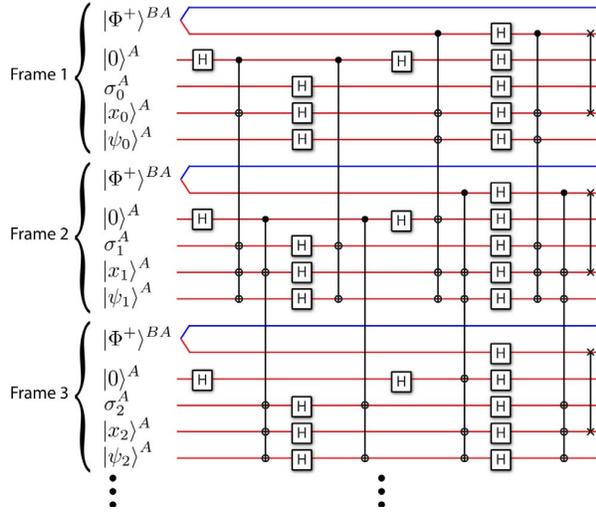
Fig. 1. The encoding circuit for the grandfather quantum convolutional code in our example. Each frame i has an ebit $|\Phi^+\rangle^{BA}$ shared between Bob and Alice, a fresh ancilla qubit $|0\rangle^A$, a gauge qubit σ_i^A, an information classical bit $|x_i\rangle^A$, and an information qubit $|\psi_i\rangle^A$. Bob's qubits are blue and Alice's qubits are red. Including the ebit and an ancilla qubit implies that the code incorporates active quantum error correction. Including the gauge qubit implies that the code has passive quantum error correction. The code provides passive error correction of phase errors on the classical bit. Alice encodes her classical bits and qubits using the encoding circuit above. The decoding circuit consists of the above operations in reverse order.

algorithm [20]. The code passively protects against errors in the group $\langle \mathcal{S}_I, \mathcal{S}_G, \mathcal{S}_C \rangle$.

Our scheme for quantum convolutional coding incorporates many of the known techniques for quantum error correction. It can take full advantage of the benefits of these different techniques.

III. EXAMPLE

We present an example of a grandfather quantum convolutional code in this section. The code protects one information qubit and one classical bit with the help of an ebit, an ancilla qubit, and a gauge qubit. The first frame of input qubits has the state

$$\rho_0 = |\Phi^+\rangle\langle\Phi^+| \otimes |0\rangle\langle0| \otimes \sigma_0 \otimes |x_0\rangle\langle x_0| \otimes |\psi_0\rangle\langle\psi_0|, \quad (15)$$

where $|\Phi^+\rangle$ is the ebit, $|0\rangle$ is the ancilla qubit, σ_0 is an arbitrary state for the gauge qubit, $|x_0\rangle$ is a classical bit represented by state $|0\rangle$ or $|1\rangle$, and $|\psi_0\rangle$ is one information qubit equal to $\alpha_0|0\rangle + \beta_0|1\rangle$. The states of the other input frames have a similar form though recall that information qubits can be entangled across multiple frames.

The initial unencoded stabilizer for the code is as follows:

$$S_0(D) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The first two rows stabilize the ebit shared between Alice and Bob. Bob possesses the half of the ebit in column one and

Alice possesses the half of the ebit in column two in both the left and right matrix. The third row stabilizes the ancilla qubit.

The generators for the initial entanglement subgroup $\mathcal{S}_{E,0}$, isotropic subgroup $\mathcal{S}_{I,0}$, gauge subgroup $\mathcal{S}_{G,0}$, and classical subgroup $\mathcal{S}_{C,0}$ are respectively as follows:

$$S_{E,0}(D) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$
$$S_{I,0}(D) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$
$$S_{G,0}(D) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$
$$S_{C,0}(D) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The sender performs the following finite-depth operations (order is from left to right and top to bottom):

$H(2)\ \ C(2,3,D)\ \ C(2,4,1+D)\ \ C(2,5,D)\ \ H(3,4,5)$
$C(2,3,D)\ \ C(2,5,D)\ \ H(2)\ \ C(1,2,D)\ \ C(1,4,1+D)$
$C(1,5,1+D)\ \ H(1,2,3,4,5)\ \ C(1,3,D)\ \ C(1,4,1+D)$
$C(1,5,1+D)\ \ S(1,4).$

Figure 1 details these operations on the initial qubit stream. The initial stabilizer matrix $S_0(D)$ transforms to $S(D) = \begin{bmatrix} Z(D) & | & X(D) \end{bmatrix}$ under these encoding operations, where

$$Z(D) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & h(D) & D & 0 & 1 & h(D) \\ 0 & 0 & 0 & D & D & D \end{bmatrix}, \quad (16)$$

$$X(D) = \begin{bmatrix} 0 & h(D) & 0 & D & 1 & h(D) \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}, \quad (17)$$

and $h(D) = 1+D$. The generators for the different subgroups transform respectively as follows:

$$S_E(D) =$$
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & h(D) & 0 & D & 1 & h(D) \\ h(D) & D & 0 & 1 & h(D) & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$S_I(D) = \begin{bmatrix} 0 & 0 & D & D & D & 0 & 1 & 0 & 1 & 1 \end{bmatrix},$$
$$S_G(D) = \begin{bmatrix} 0 & \frac{1}{D} & 1 & \frac{1}{D} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{D} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$S_C(D) =$$
$$\begin{bmatrix} 1 & h\left(\frac{1}{D}\right) & 0 & h\left(\frac{1}{D}\right) & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The code actively protects against an arbitrary single-qubit error in every other frame. One can check that the syndromes of the stabilizer in $S(D)$ satisfy this property. Consider the Pauli generators corresponding to the generators in the entanglement subgroup and the isotropic subgroup:

$$\cdots \begin{vmatrix} X & I & I & X & X & X & I & X & I & X \\ Z & I & I & Z & Z & Z & Z & I & I & Z \\ I & X & I & X & X & I & I & Z & Z & Z \end{vmatrix} \cdots, \quad (18)$$

where all other entries in the left and right directions are tensor products of the identity. We can use a table-lookup syndrome-based algorithm to determine the error-correcting

Error	Syndrome	Error	Syndrome	Error	Syndrome
X_1	001100	X_3	000001	X_5	001101
Y_1	111100	Y_3	010001	Y_5	111111
Z_1	110000	Z_3	010000	Z_5	110010
X_2	000100	X_4	001001		
Y_2	000110	Y_4	101011		
Z_2	000010	Z_4	100010		

TABLE I

A LIST OF POSSIBLE SINGLE-QUBIT ERRORS IN A PARTICULAR FRAME AND THE CORRESPONDING SYNDROME VECTOR. THE SYNDROME CORESPONDING TO ANY SINGLE-QUBIT ERROR IS UNIQUE. THE CODE THEREFORE CORRECTS AN ARBITRARY SINGLE-QUBIT ERROR IN EVERY OTHER FRAME.

capability of the code. The method is similar to the technique originally outlined in detail in Ref. [16]. The syndrome vector s consists of six bits where $s = s_1 \cdots s_6$. The first bit s_1 is one if the error anticommutes with the operator $XIIXX$ in the first part of the first generator above and zero otherwise. The second bit s_2 is one if the error anticommutes with the operator $XIXIX$ in the delayed part of the first generator above and zero otherwise. The third through sixth bits follow a similar pattern for the second and third generators above. Table I lists all single-qubit errors over five qubits and their corresponding syndromes. The code corrects an arbitrary single-qubit error in every other frame using this algorithm because the syndromes are all unique. A syndrome-based Viterbi algorithm might achieve better performance than the simple syndrome table-lookup algorithm outlined above.

This code also has passive protection against errors in $\langle \mathcal{S}_I, \mathcal{S}_G, \mathcal{S}_C \rangle$. The Pauli form of the errors in this group span over three frames and are as follows:

$$\cdots \begin{vmatrix} IIIII \\ IZIZI \\ IZIII \\ IZIZI \end{vmatrix} \begin{vmatrix} IXIXX \\ IIZII \\ IIXII \\ ZZIZI \end{vmatrix} \begin{vmatrix} IIZZZ \\ IIIII \\ IIIII \\ IIIII \end{vmatrix} \cdots \quad (19)$$

The smallest weight errors in this group have weight two and three. The code passively corrects the above errors or any product of them or any five-qubit shift of them.

There is a trade-off between passive error correction and the ability to encode quantum information as discussed in Ref. [11]. One can encode more quantum information by dropping the gauge group and instead encoding an extra qubit. The gauge generators then become logical X and Z operators for the extra encoded qubits. One can also turn the classical bit into a qubit by dropping the generators in the classical subgroup. These generators then become logical Z operators for the extra encoded qubits.

IV. CONCLUSION

We have presented a framework and a representative example for grandfather quantum convolutional codes. We have explicitly shown how these codes operate, and how to encode and decode a classical-quantum information stream by using ebits, ancilla qubits, and gauge qubits for redundancy. The ultimate goal for this theory is to find quantum convolutional codes that might play an integral part in larger quantum codes that approach the grandfather capacity [8]. One useful line of investigation may be to combine this theory with the recent quantum turbo-coding theory [21].

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