

Quantum rate distortion, reverse Shannon theorems, and source-channel separation

Nilanjana Datta, Min-Hsiu Hsieh, Mark Wilde

- (1) University of Cambridge, U.K.
- (2) McGill University, Montreal, Canada

Classical Information Theory



Shannon's Source
Coding Theorem

*Compression of information
emitted by a source*

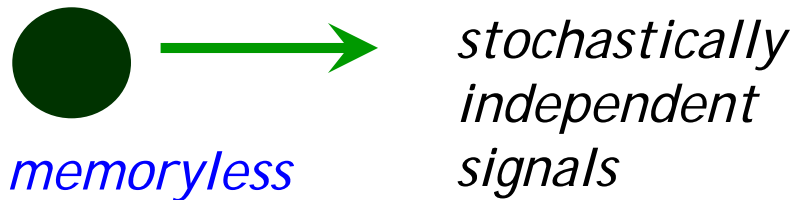


Shannon's Noisy Channel
Coding Theorem

*Transmission of information
through a noisy channel*

Shannon's Coding Theorems

■ Source Coding Theorem



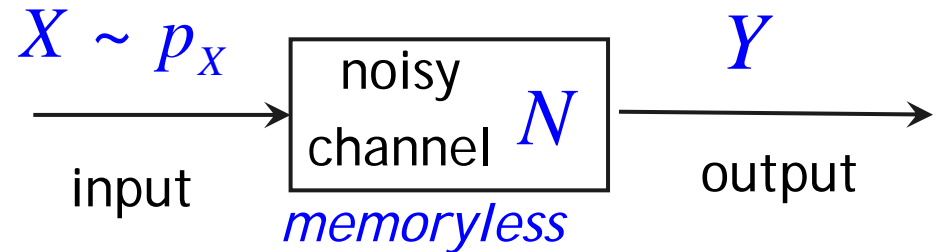
$$X \sim p_X \quad ; \quad x \in X$$

The fundamental limit of data compression:

= *Shannon entropy* of the source:

$$H(X)$$

■ Channel Coding Theorem



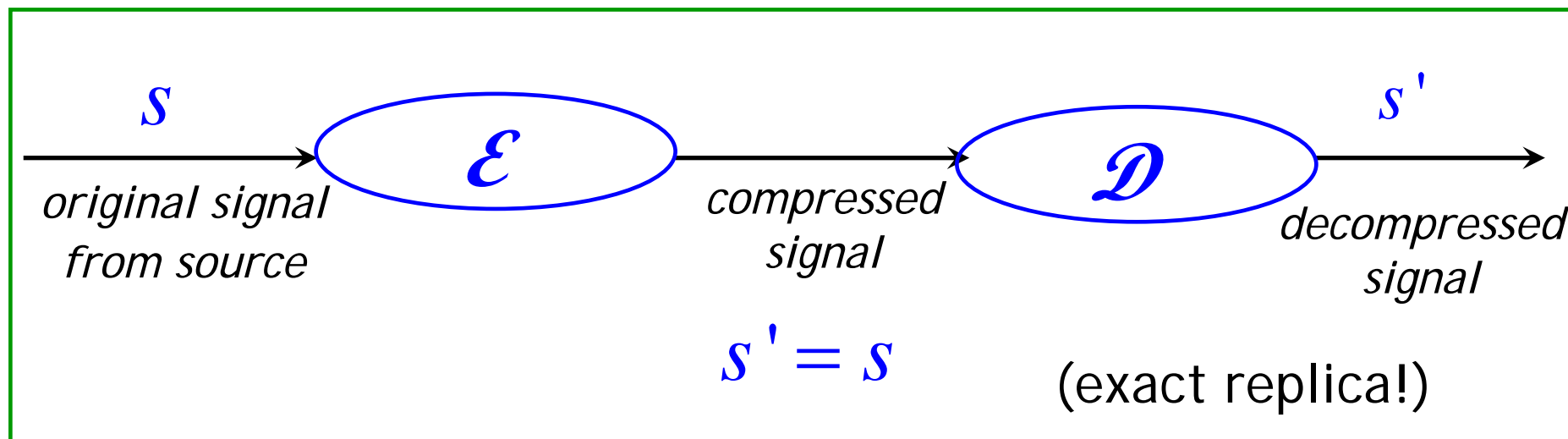
The fundamental limit on the rate of reliable information transmission:

= *Capacity* of the channel

$$C(N) = \max_{p_X} I(X : Y)$$

mutual information

Lossless Data Compression



Shannon's Source Coding Theorem

corresponds to asymptotically lossless data compression

- multiple (n) uses of the channel
- **signals:** sequences x_1, x_2, \dots, x_n

$$\bar{p}_e(n) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty$$

- (average probability of error in recovering original signal)

Lossless Compression - often too **stringent** a condition

- for cases of **multimedia data**;
e.g. audio, video and still images
- when **storage space** is **insufficient**

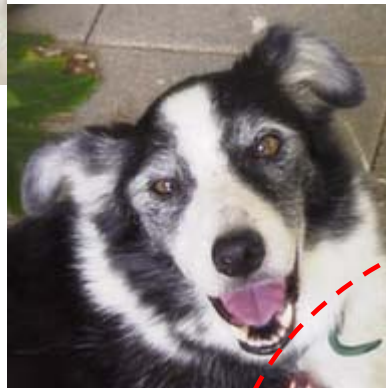
- **Why ?**

Typically, a substantial amount of **data** can be
discarded before the information is
sufficiently degraded to be **noticeable!**

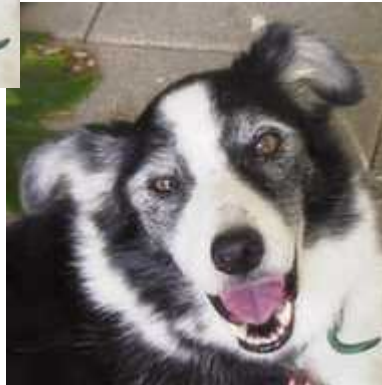
An Example



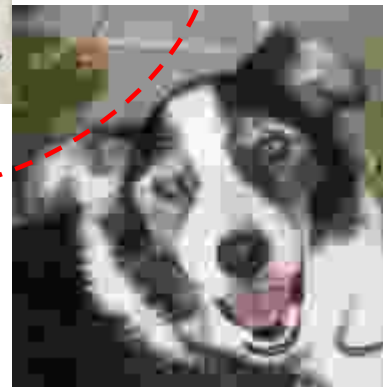
original image (uncompressed), size 108.5 KB



*lossless compression (PNG);
size 60.1 KB*



lossy compression; 4.82 KB



lossy compression; 1.14 KB

Lossy Data Compression

- a data compression scheme :

■ recovered data \neq original data

instead

■ recovered data $\stackrel{D}{\approx}$ original data

D : the allowed distortion

Rate Distortion Theory = the theory of **Lossy Data Compression**

- rate of compression $\overset{\text{tradeoff}}{\longleftrightarrow}$ the allowed distortion



fundamental limit on the asymptotic rate of data compression for a given maximum distortion

Classical Rate Distortion Theory (Shannon)

- For a given memoryless information source ; $X \sim p_X$
- If the maximum allowed distortion is D ; $0 \leq D < 1$,

The minimum rate of data compression:

$$R(D) = \min_{(a)} I(X:Y) \leq H(X)$$

rate distortion function

mutual information

$$(a) : p_{Y|X} ; \mathbf{E}(d(X, Y)) \leq D$$

stochastic maps

average distortion

Y : output of a stochastic map $p_{Y|X} : X \rightarrow Y$

$d(x, y)$: distortion measure e.g. $d(x, y) = (x - y)^2$

Our Aim

To obtain **rate distortion functions** in the **quantum realm**

Scenario for Quantum Rate Distortion

for a fixed $0 \leq D < 1$

- Storage setting



$\{\rho, \mathcal{H}_A\}$

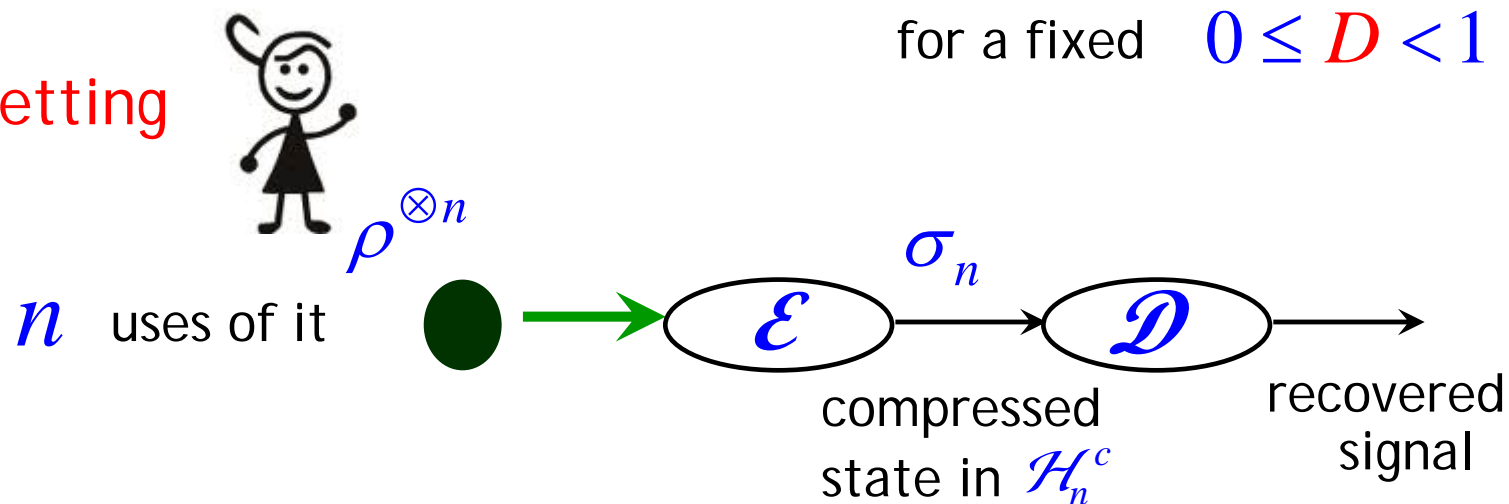
a
memoryless
quantum
information
source



Scenario for Quantum Rate Distortion

for a fixed $0 \leq D < 1$

- Storage setting



Rate of data compression = R if $\dim \mathcal{H}_n^c = 2^{nR}$

achievable - if $\lim_{n \rightarrow \infty} \bar{d}(\rho^{\otimes n}, \mathcal{D} \circ \mathcal{E}) \leq D$

$$R_q(D) := \inf\{R : \text{achievable}\}$$

Quantum Rate Distortion function

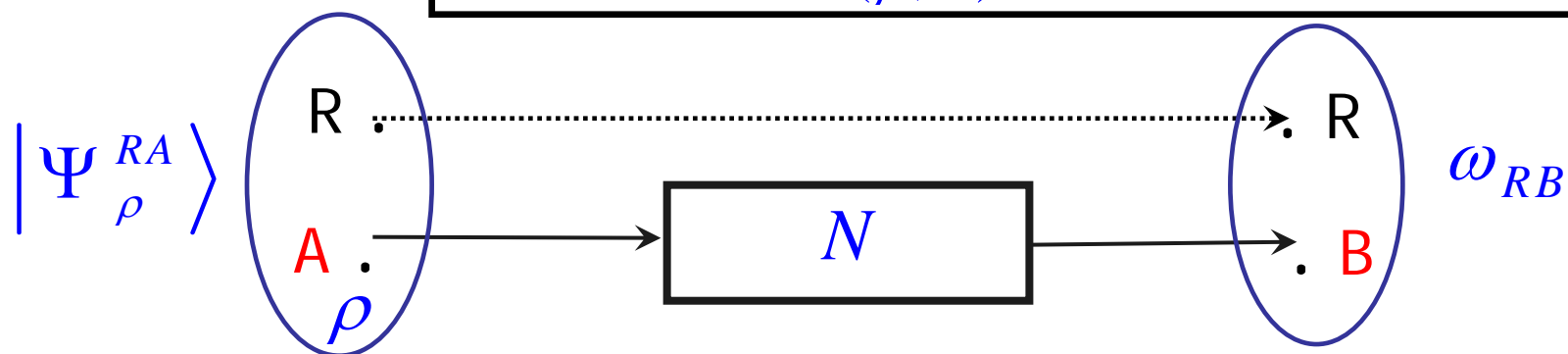
(min. rate of compression
under given distortion)

Quantum Rate Distortion Function

Barnum proved:

$$R_q(D) \geq \min_{\substack{N: \text{CPTP} \\ d(\rho, N) \leq D}} I_c(\rho, N) \dots\dots(1)$$

coherent information



$$I_c(\rho, N) = -S(R|B)_\omega; \quad d(\rho, N) = 1 - F_e(\rho, N)$$

entanglement fidelity

■ He conjectured: " $=$ " holds in (1)

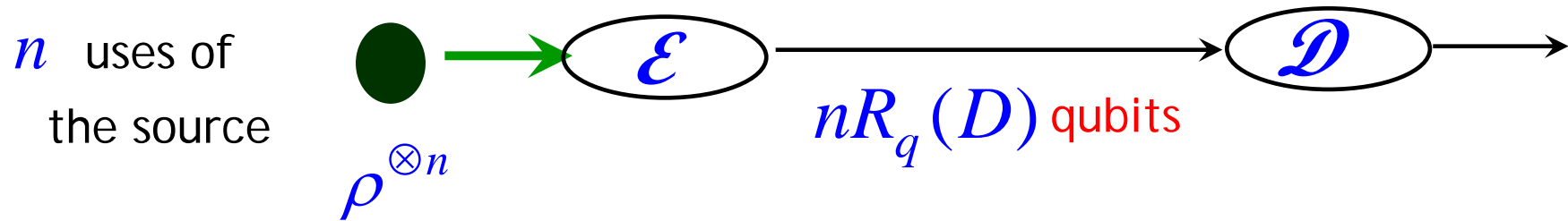
problem!!

- the coherent information can be negative
- BUT
- The rate of a data compression scheme is non-negative !!

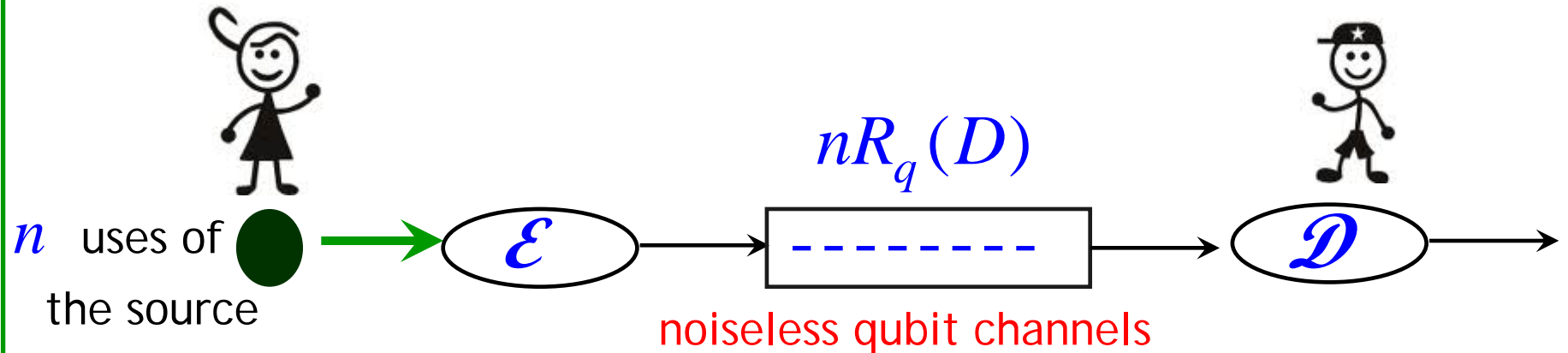
Equivalent Scenarios for Quantum Rate Distortion



■ Storage setting:

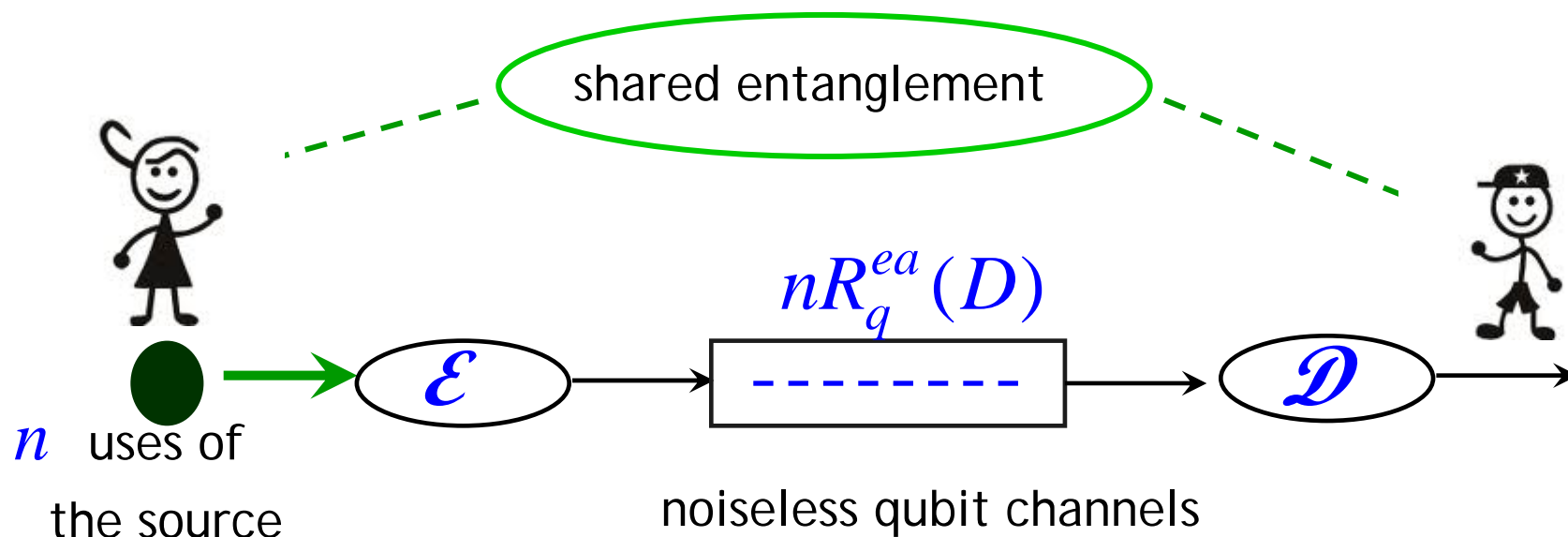


■ Communication setting:



Another Scenario for Quantum Rate Distortion

- Communication setting: (entanglement assisted)



$$R_q^{ea}(D) = \text{min. number of noiseless qubit channels needed per use of the source in the presence of entanglement for a given distortion } D$$

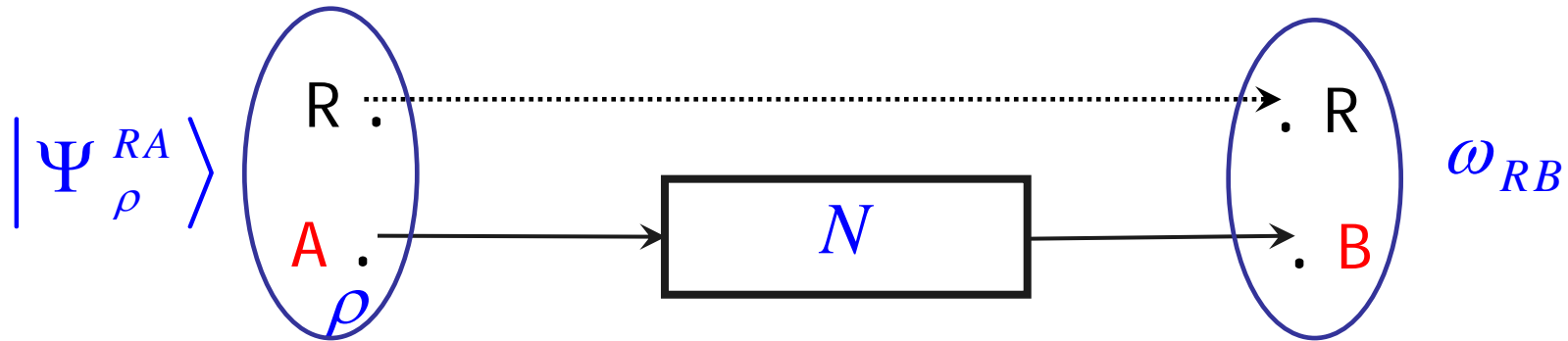
entanglement-assisted quantum rate distortion function

Result - I

- Entanglement-assisted quantum rate distortion function:

$$R_q^{ea}(D) = \min_{\substack{N: \text{CPTP} \\ d(\rho, N) \leq D}} \frac{1}{2} I(R:B)_{\omega} \dots\dots\dots (2)$$

quantum mutual information



- Proof:

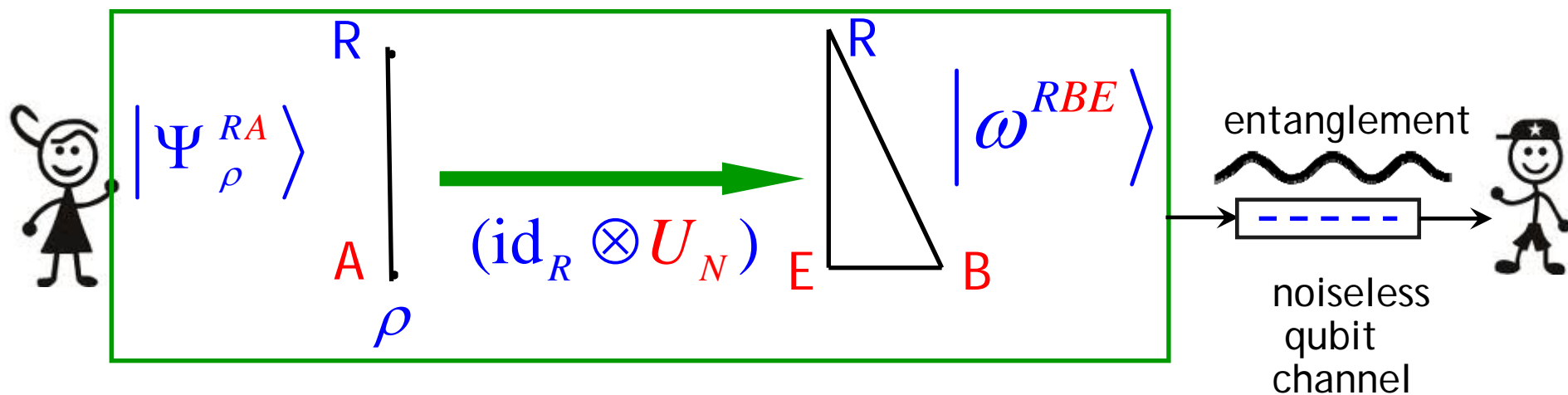
(i) Prove : $R_q^{ea}(D) \geq$ RHS of (2) *using entropic inequalities*

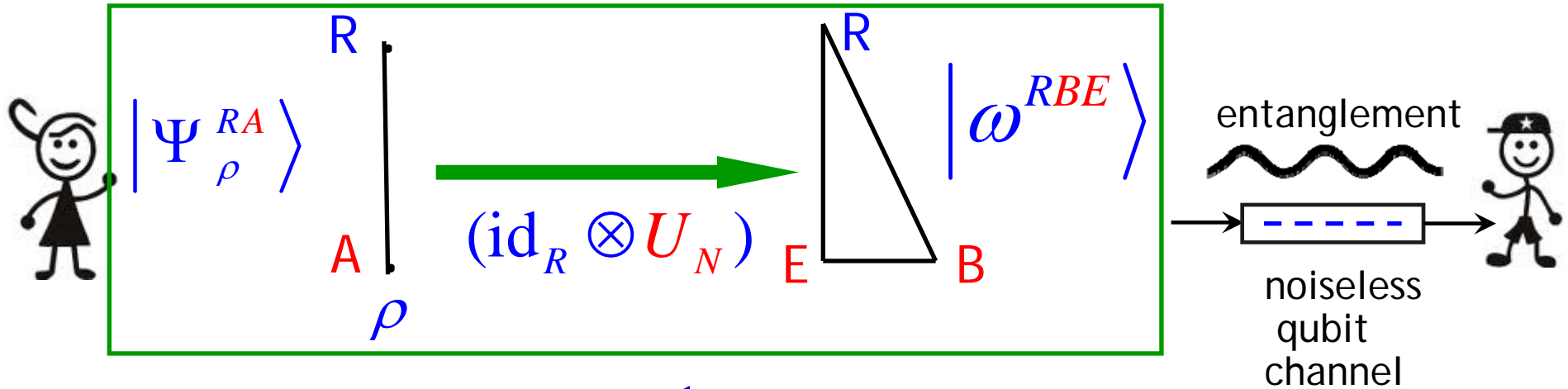
(ii) Prove: This bound is **achievable**, i.e., " $=$ " holds

■ The rate $R = \min_{\substack{N: \text{CPTP} \\ d(\rho, N) \leq D}} \frac{1}{2} I(R: B)_\omega$ is **achievable**

Sketch of proof:

- Source $\{\rho, \mathcal{H}_A\}$; Its purification: $|\Psi_\rho^{RA}\rangle$
- Let $N =$ the **minimizing** CPTP map
- Let $U_N : A \rightarrow BE$ a Stinespring isometry of N





To prove: $R = \min_{\substack{N: \text{CPTP} \\ d(\rho, N) \leq D}} \frac{1}{2} I(R : B)_\omega$ is achievable

■ Note:

$$\omega^B := N(\rho) \text{ satisfies } d(\rho, N) \leq D$$

Suffices to prove:

ω_B can be sent to Bob by Alice through $\frac{1}{2} I(R : B)_\omega$ uses of a noiseless qubit channel per use of the source

- How can we transfer ω_B to Bob ?

- initially



$$\rho^{\otimes n}$$

$$\xrightarrow{U_N^{\otimes n}}$$

$$\omega_{BE}^{\otimes n}$$

Asymptotic setting

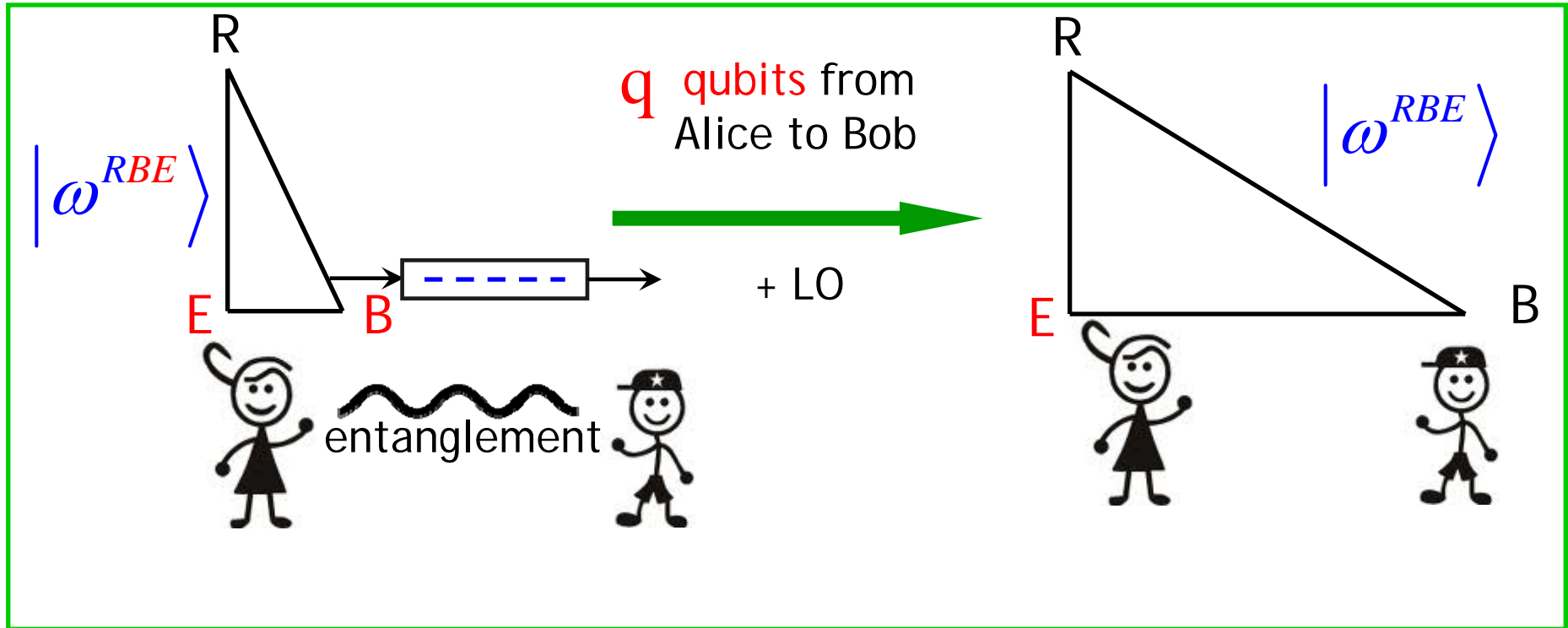
*State
splitting*

- finally



$$\sigma_n \approx \omega_B^{\otimes n}$$

- How can we transfer ω_B to Bob ?



$$q = \frac{1}{2} I(R : B)_\omega = \min_{\substack{N: \text{CPTP} \\ d(\rho, N) \leq D}} \frac{1}{2} I(R : B)_\omega$$

achievable rate !

- (i) **local preparation** of $\rho^{\otimes n}$ $\xrightarrow{U_N^{\otimes n}}$ $\omega_{BE}^{\otimes n}$
- (ii) **State splitting** with the help
of **shared entanglement** such that $\omega_B^{\otimes n}$



≡ **simulating** the (output of the) **quantum channel** $N^{\otimes n}$
when the input is $\rho^{\otimes n}$ (using shared entanglement)

= a **special case** of another protocol -- **channel simulation**
- **Quantum Reverse Shannon Theorem (QRST)**

⇒ achievability of $R_q^{ea}(D)$

Result - II

- **Unassisted** Quantum rate distortion function:

$$R_q(D) = \lim_{n \rightarrow \infty} \frac{1}{n} \min_{\substack{N^{(n)}: \text{CPTP} \\ d(\rho^{\otimes n}, N^{(n)}) \leq D}} E_P^\infty(N^{(n)}(\rho^{\otimes n}))$$

regularized entanglement of purification

- For any $\sigma_{AB} \rightarrow |\sigma_{ABR}\rangle \rightarrow \sigma_{BR}$

$$E_P(\sigma_{BR}) := \min_{\Lambda_R: \text{CPTP}} H((\text{id}_B \otimes \Lambda_R)\sigma_{BR}) \geq 0$$

- where,
for any state ρ , $H(\rho)$: Von Neumann entropy of ρ

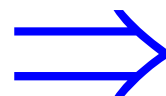
Result - II

- **Unassisted** Quantum rate distortion

function:

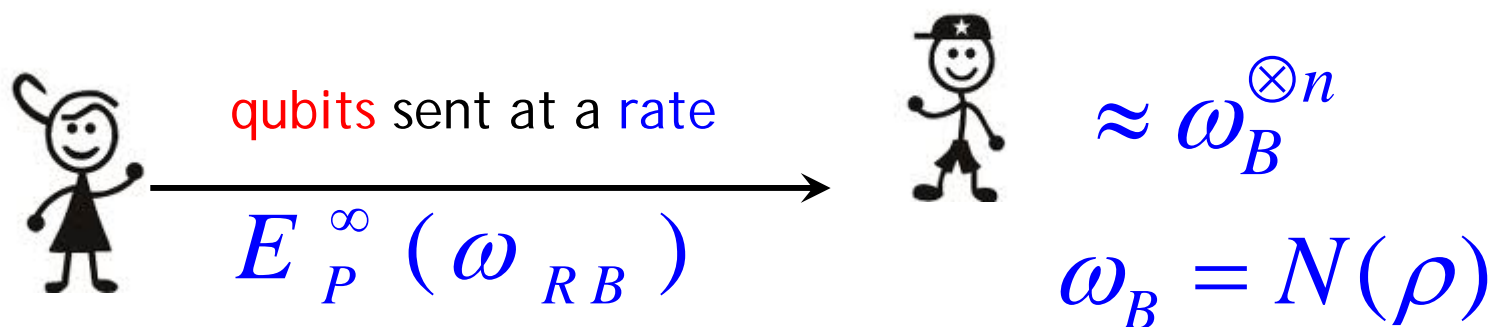
$$R_q(D) = \lim_{n \rightarrow \infty} \frac{1}{n} \min_{\substack{N: \text{CPTP} \\ d(\rho, N^{(n)}) \leq D}} E_P^\infty \left(N^{(n)}(\rho^{\otimes n}) \right)$$

Channel simulation (QRST) in the
absence of **shared entanglement**



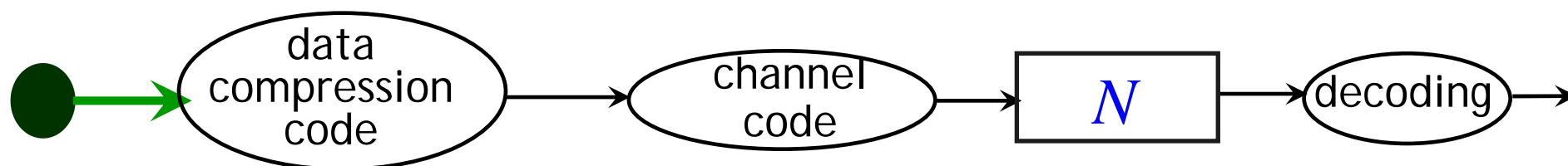
achievability of
this rate

To simulate output of $N^{\otimes n}$ when input is $\rho^{\otimes n}$



Source Channel Separation Theorems

- Shannon: connects the two pillars of Classical Info. Theory
- Is it possible to **transmit** a classical information **source** (X) reliably over a classical **channel** (N) ?
- Yes -- **if & only if** $H(X) \leq C(N)$
- Implication:



- a **2-stage encoding & decoding** with the best data compression + error correction codes is **optimal!**

Result - III

Quantum Source Channel Separation Theorems

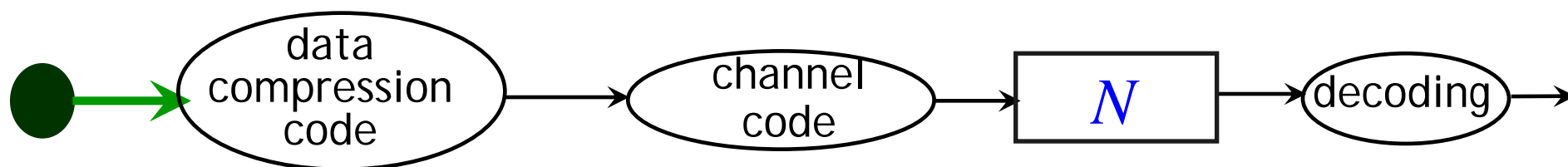
Theorem : It possible to transmit a quantum information source (ρ) reliably over a quantum channel (N) if & only if

von Neumann entropy

$$H(\rho) \leq Q(N)$$

quantum capacity

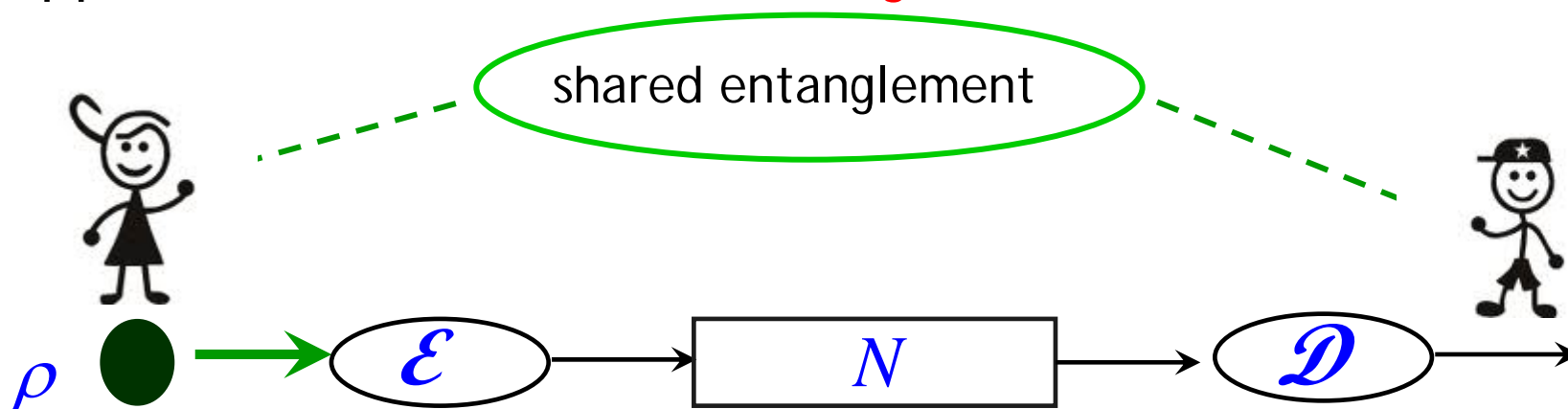
■ Implication:



- a 2-stage encoding & decoding with the best data compression + quantum error correction codes is optimal!

What if $H(\rho) > Q(N)$?

- Suppose Alice & Bob share entanglement, i.e.,



- Theorem: it is possible to transmit the source ρ over the channel N , up to some distortion D

if & only if

$$R_q^{ea}(D) \leq \frac{1}{2} I_c(\rho, N)$$

Summary

- $R_q^{ea}(D)$: ■ entanglement-assisted quantum rate distortion function
- in terms of the **quantum mutual information**

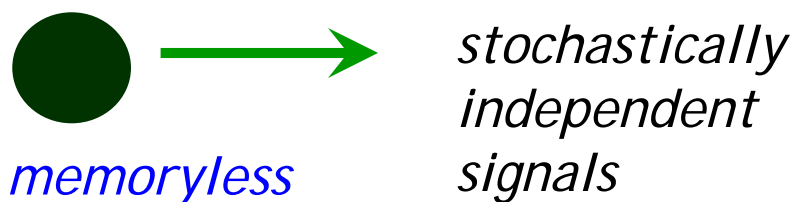
- $R_q(D)$: ■ unassisted quantum rate distortion function
- in terms of regularised **entanglement of purification**

- Quantum Source Channel Separation Theorems

- *The following are some extra slides which I have removed/modified to make the talk 20 minute long.*

Shannon's Coding Theorems

■ Source Coding Theorem



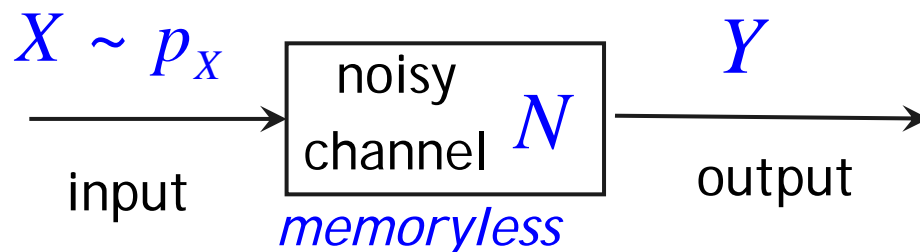
$$X \sim p_X \quad ; \quad x \in X$$

The fundamental limit of data compression:

= *Shannon entropy* of the source:

$$H(X)$$

■ Channel Coding Theorem



The fundamental limit on the rate of reliable information transmission:

= *Capacity* of the channel

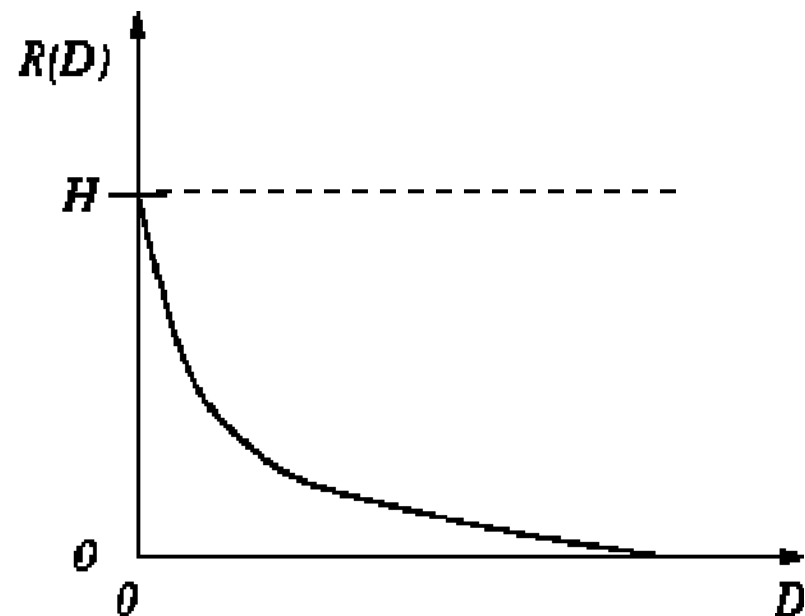
$$C(N) = \max_{p_X} I(X : Y)$$

mutual information

Rate Distortion Function

$$R(D) = \min_{\substack{p_{Y|X}: \\ E(d(X,Y)) \leq D}} I(X : Y)$$

- For $D = 0$, $R(D) = H(X)$
 $Y = X$
- For $D > 0$, $R(D) < H(X)$



Barnum's Conjecture

Quantum Rate Distortion
function

$$R_q(D) = \min_{\substack{N: \text{CPTP} \\ d(\rho, N) \leq D}} I_c(\rho, N)$$

■ Motivation :

← similarity in the **classical case**

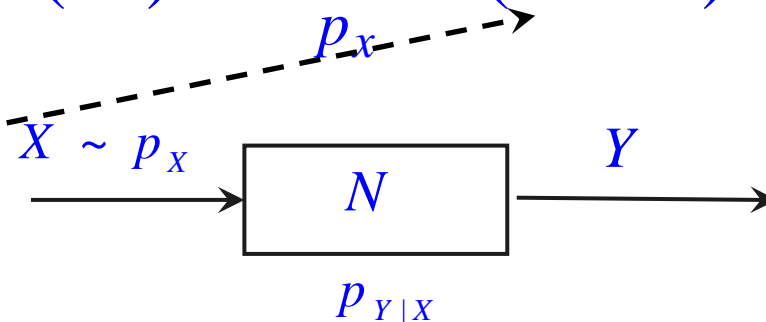
Rate Distortion Function

$$R(D) = \min_{\substack{p_{Y|X}(y|x): \\ E(d(x,y)) \leq D}} I(X : Y)$$

mutual information

Capacity of a noisy channel

$$C(N) = \max_{p_X} I(X : Y)$$

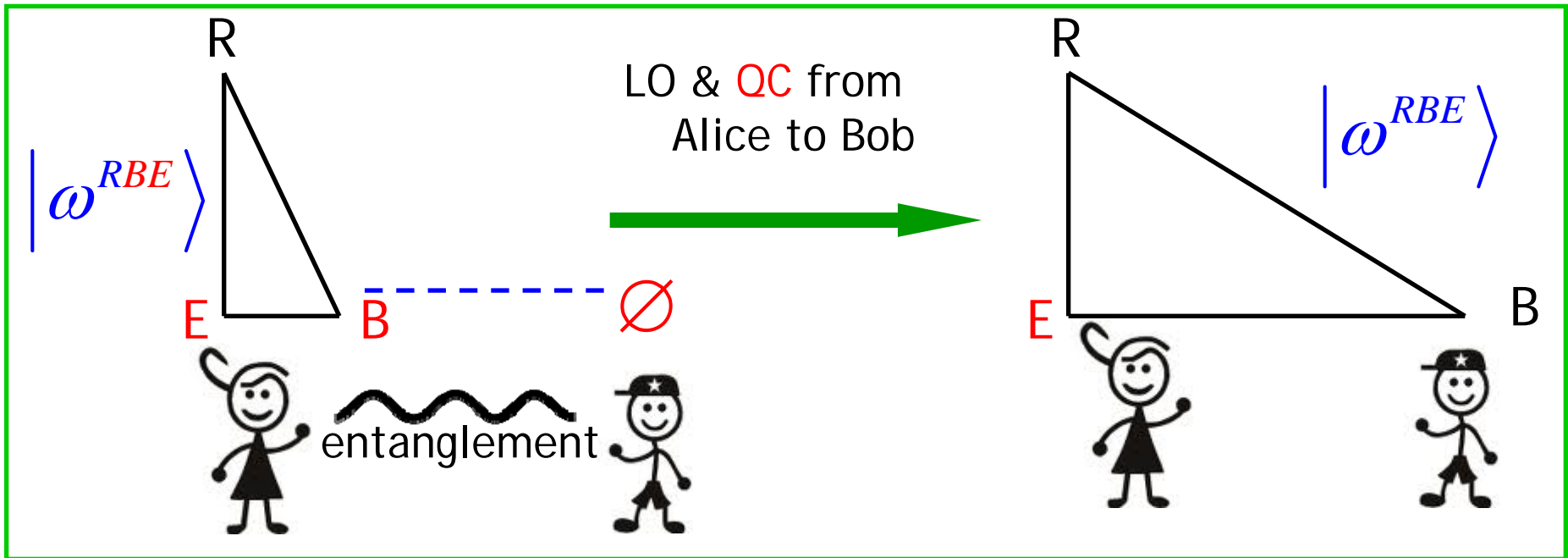


Quantum capacity of a
noisy **quantum** channel

$$Q(N) = \lim_{n \rightarrow \infty} \frac{1}{n} \max_{\rho} I_c(\rho, N^{\otimes n})$$

coherent information

State Splitting



Resource Inequality : $\omega^{BE|\emptyset} + \mathbf{q} [q \rightarrow q] + \mathbf{e} [qq]_{\infty} \geq \omega^{E|B}$

$$\mathbf{e} = \frac{1}{2} I(B : E)_{\omega} ;$$

$$\mathbf{q} = \frac{1}{2} I(R : B)_{\omega} = \min_{\substack{N: \text{CPTP} \\ d(\rho, N) \leq D}} \frac{1}{2} I(R : B)_{\omega}$$

achievable rate !

Result - II

- **Unassisted** Quantum rate distortion

function:

$$R_q(D) = \lim_{n \rightarrow \infty} \frac{1}{n} \min_{\substack{N: \text{CPTP} \\ d(\rho, N^{(n)}) \leq D}} E_P^\infty \left(N^{(n)}(\rho^{\otimes n}) \right)$$

Channel simulation (QRST) in the
absence of **shared entanglement**



achievability of
this rate

To simulate output of $N^{\otimes n}$ when input is $\rho^{\otimes n}$

Resource Inequality :

$$E_P^\infty(\omega_{RB}) [q \rightarrow q] \underset{\infty}{\geq} \omega_B$$



$$E_P^\infty(\omega_{RB})$$



qubits



$$\omega_B = N(\rho)$$

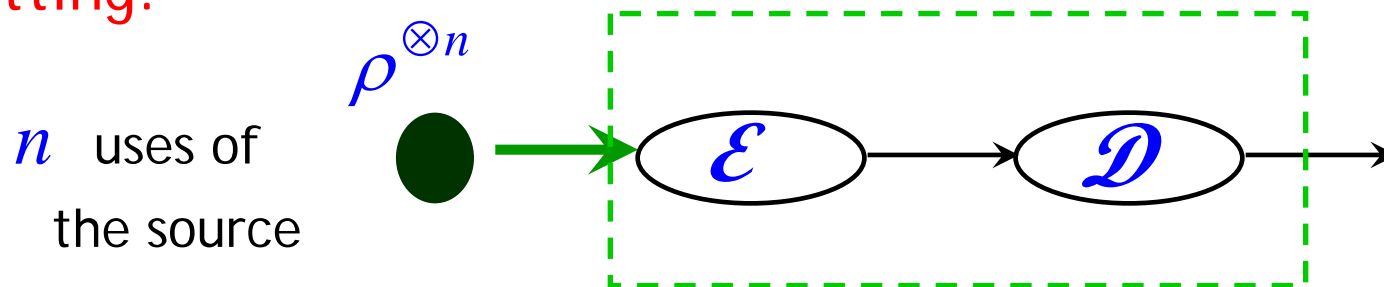
- Unassisted Quantum rate distortion function:

$$R_q(D) = \lim_{n \rightarrow \infty} \frac{1}{n} \min_{\substack{N: \text{CPTP} \\ d(\rho, N^{(n)}) \leq D}} E_P^\infty(N^{(n)}(\rho^{\otimes n}))$$

- Why the double regularization?



- Storage setting:



- whereas in QRST: simulates $N^{\otimes n}$

$$\mathcal{D} \circ \mathcal{E} \neq N^{\otimes n}$$

- To prove " = " need to prove the converse
 - done using entropic inequalities