The Quest for a Quantum Simultaneous Decoder

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arXiv:1102.2624

Difficult Problems in Quantum Information Theory, MIT, May 4, 2011

Motivation

Question: Why would you need this conjecture when **Andreas Winter** found the capacity of the quantum MAC channel as a *graduate student*?

quant-ph/9807019

Answer: The best known achievable rate region for the **classical interference channel** requires **simultaneous decoding** and we would like to "quantize" this technique.
It would also solve many other *difficult problems in quantum information theory :)*

This conjecture will need to be proved
in order to have a characterization of the
capacity of the **quantum multiple access channel**
in **one-shot information theory**
(Time-sharing is NOT an option!)

Overview

- Review of **Typical Sequences**
- **Simultaneous Decoder** in Classical Info. Theory
- Classical Coding for Quantum Channels Overview
- Quantum **Multiple Access** Channel
- **Quantum Simultaneous Decoder** Conjecture
- Attempts at Proving the Conjecture

Review of Typical Sequences

A sequence x^n emitted from many **IID realizations** of a random variable $X \sim p_X(x)$ is *typical* if its **sample entropy** is close to its **true entropy**:

$$T_\delta^{X^n} \equiv \{x^n : \left|\overline{H}(x^n) - H(X)\right| \leq \delta\}$$

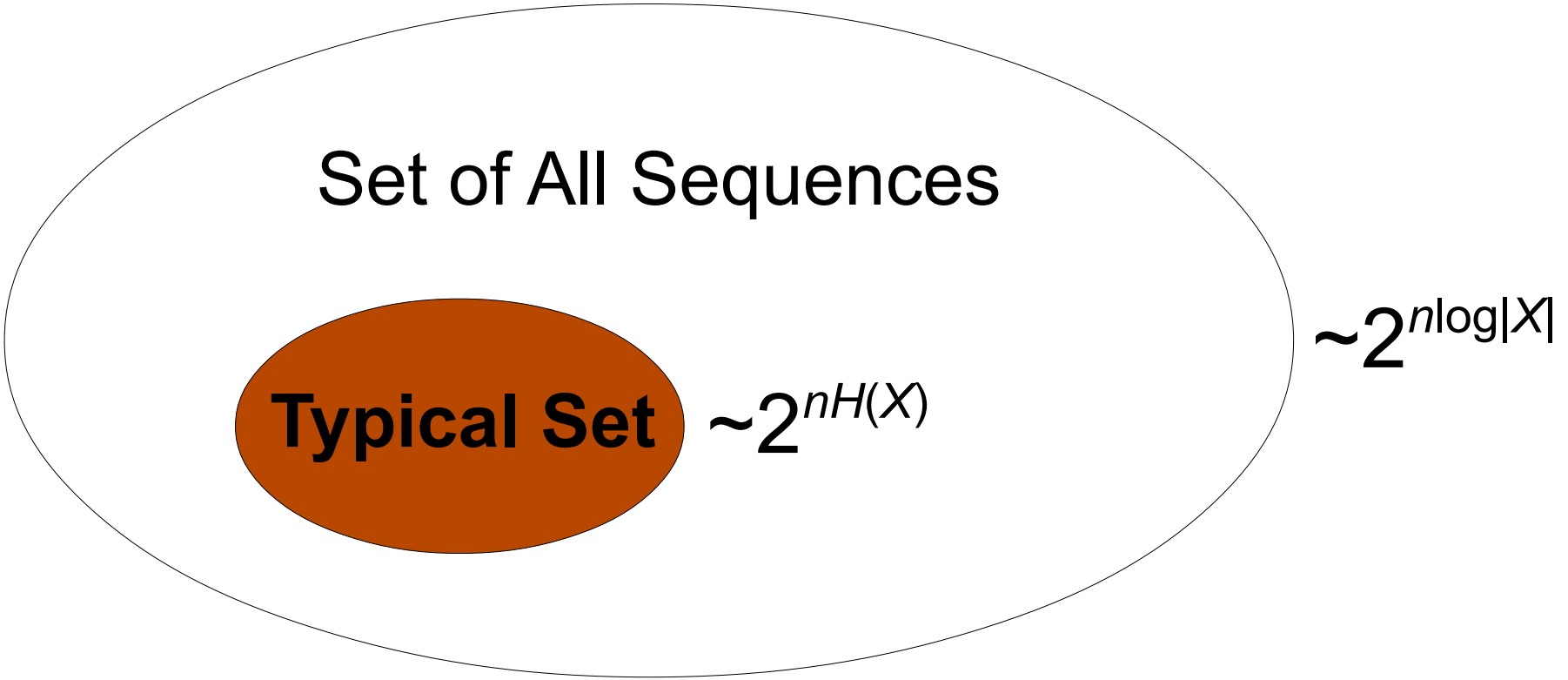
where

$$\overline{H}(x^n) \equiv -\frac{1}{n}\log(p_{X^n}(x^n))$$

$$H(X) \equiv -\sum_x p_X(x)\log(p_X(x))$$

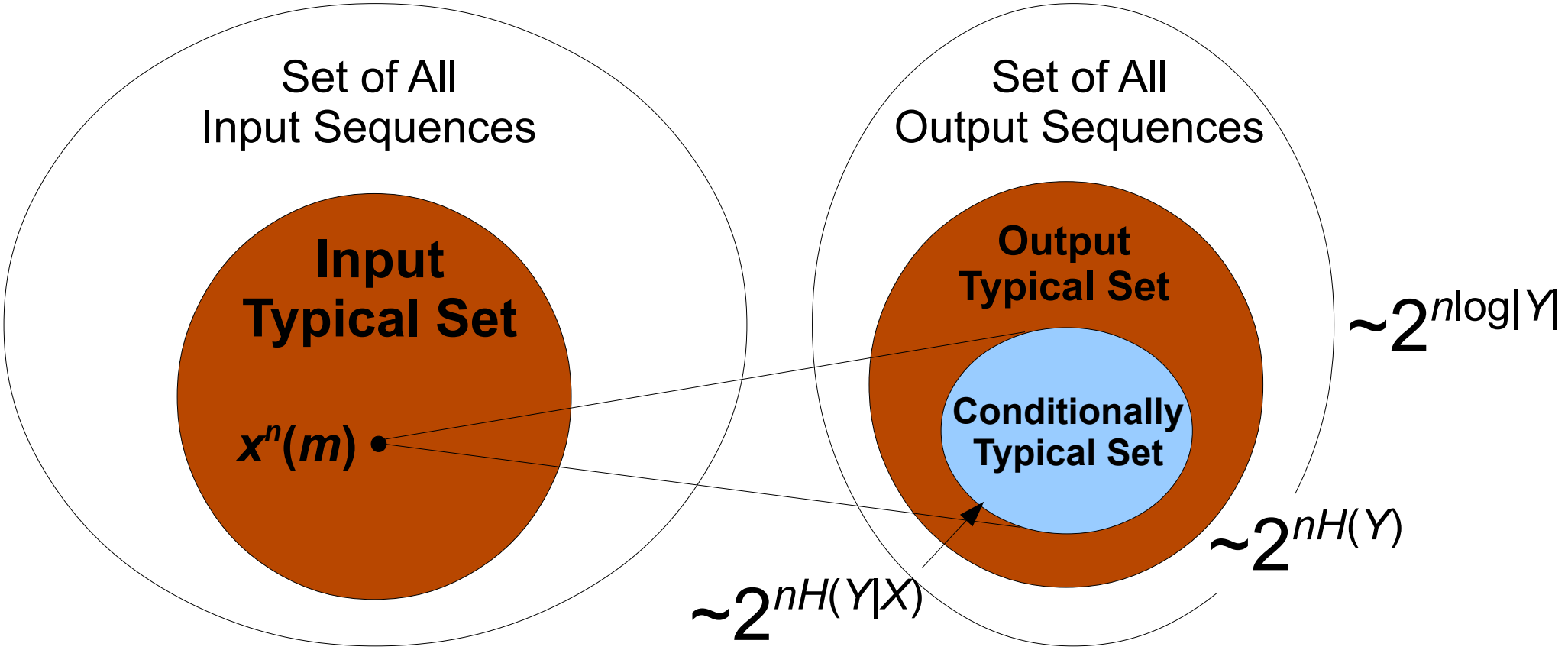
Key Property of Typical Sequences

The typical set ***asymptotically*** has
all of the probability, while being
exponentially smaller
than the set of all sequences:



Random Coding over Noisy Channels

- Choose codewords $\{x^n(m)\}_m$ **randomly** from typical set for X^n
- Send codeword $x^n(m)$ over **IID** channel $p_{Y|X}(y|x)$
- Receiver exploits **jointly typical decoding**



Error Analysis

A **jointly typical decoder** declares message *m* was sent if it is the unique message such that

$$(x^n(m), y^n) \in T_\delta^{X^n Y^n}$$

Shannon's idea was to analyze the ***expectation of the average error probability***

From **symmetry** of code construction, it suffices to analyze error probability for **first message**

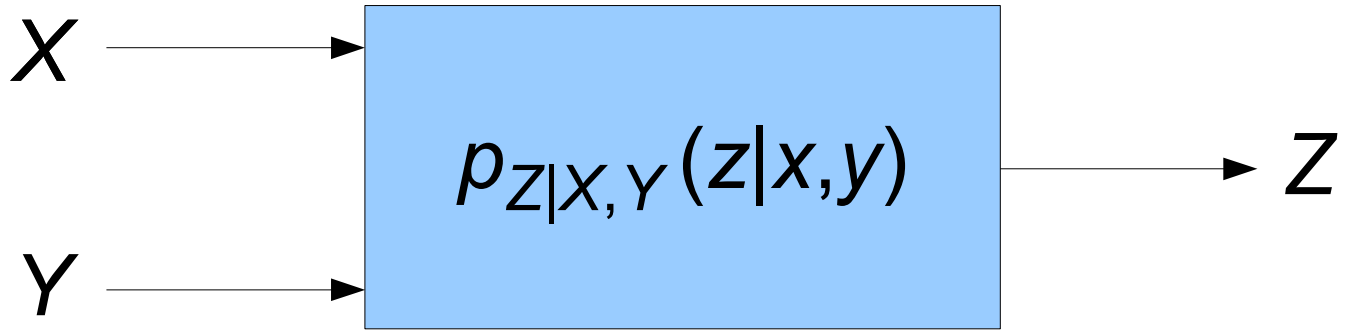
Two kinds of error can occur:

1) Sequences $x^n(1)$ and y^n are not jointly typical. Bound this by ε from LLN

2) Some other codeword $x^n(m')$ is jointly typical with y^n. Gives packing bound: $|\mathcal{M}|\, 2^{-nI(X;Y)}$

Multiple Access Channel (MAC)

Two senders and one receiver:



- Choose codewords $\{x^n(l)\}_l$ and $\{y^n(m)\}_m$ **randomly** from typical set for X^n and Y^n, respectively
- Send codewords $x^n(l)$ and $y^n(m)$ over **IID** channel $p_{Z|X,Y}(z|x,y)$
- Receiver exploits **simultaneous decoding**

Error Analysis for MAC

A **simultaneous decoder** declares that messages *l* and *m* were sent if they are the unique messages such that

$$(x^n(l), y^n(m), z^n) \in T_\delta^{X^n Y^n Z^n}$$

Four kinds of error can occur:

1) Sequences $x^n(1)$, $y^n(1)$, and z^n are not jointly typical. Bound this by ε from LLN

2) $x^n(1)$ typical but some other codeword $y^n(m')$ is jointly typical with z^n: $|\mathcal{M}|\, 2^{-nI(Y;ZX)}$

3) $y^n(1)$ typical but some other codeword $x^n(l')$ is jointly typical with z^n: $|\mathcal{L}|\, 2^{-nI(X;ZY)}$

4) Some other codewords $x^n(l')$ and $y^n(m')$ are jointly typical with z^n: $|\mathcal{L}|\,|\mathcal{M}|\, 2^{-nI(XY;Z)}$

Classical Comm. Over Quantum Channels

For simplicity, let's study a channel
with **classical input** and **quantum output**:

$$x \longrightarrow \rho_x$$

If we use **random coding** according to $p_X(x)$ at input,
then **expected density operator** at output is

$$\rho \equiv \sum_x p_X(x) \rho_x$$

Can still use **random coding**
to select codebook $\{x^n(m)\}_m$,
but need to design a **decoding POVM**

Holevo-Schumacher-Westmoreland Decoding POVM

Consider spectral decomposition: $\rho = \sum_z \lambda_z |z\rangle\langle z|$

Use **typical projector** for expected state:

$$\Pi_{\rho,\delta}^n \equiv \sum_{z^n \in T_\delta^{Z^n}} |z^n\rangle\langle z^n|$$

And **conditionally typical projectors** for codeword states:

$$\Pi_{\rho_{x^n},\delta} \equiv \bigotimes_x \Pi_{\rho_x,\delta}$$

To form the elements of the **HSW decoding POVM**

$$\Lambda_m \equiv \left(\sum_{m'} \Pi_\rho \Pi_{\rho_{x^n(m')}} \Pi_\rho \right)^{-1/2} \Pi_\rho \Pi_{\rho_{x^n(m)}} \Pi_\rho \left(\sum_{m'} \Pi_\rho \Pi_{\rho_{x^n(m')}} \Pi_\rho \right)^{-1/2}$$

Error Analysis

Analyze **expectation of avg. error probability** for first message (exploit Shannon's idea):

$$\mathbb{E}_{X^n}\left[\mathrm{Tr}\left\{(I-\Lambda_1)\rho_{x^n(1)}\right\}\right]$$

Exploit **Hayashi-Nagaoka operator inequality** to bound the error from above by two terms:

1) Probability that codeword is not typical:

$$\mathrm{Tr}\left\{\left(I-\Pi_{\rho_{x^n(1)},\delta}\right)\rho_{x^n(1)}\right\}\leq\epsilon$$

2) Probability that some other codeword is typical:

$$\mathbb{E}_{X^n}\left[\sum_{m'\neq 1}\mathrm{Tr}\left\{\Pi_{\rho_{x^n(m')}}\Pi_\rho\rho_{x^n(1)}\Pi_\rho\right\}\right]\leq|\mathcal{M}|\,2^{-nI(X;B)}$$

Quantum Multiple Access Channel

Two **spatially separated** senders and one receiver.

Two **classical inputs** and one **quantum output**:

$$x, y \longrightarrow \rho_{x,y}$$

- Choose codewords $\{x^n(l)\}_l$ and $\{y^n(m)\}_m$ **randomly** from typical set for X^n and Y^n, respectively
- Send codewords $x^n(l)$ and $y^n(m)$ over **IID** quantum MAC
- Receiver *would like* to exploit **simultaneous decoding**

Decoding POVM for Quantum MAC

How to construct it when $\rho_{x,y}$ **do not commute**?

To start, let us first suppose they *do* commute

Define the **density operators**:

$$\rho_x \equiv \sum_y p_Y(y)\rho_{x,y}$$

$$\rho_y \equiv \sum_x p_X(x)\rho_{x,y}$$

$$\rho \equiv \sum_{x,y} p_X(x)p_Y(y)\rho_{x,y}$$

And their **typical projectors**:

$$\Pi_{\rho_x{}^n,\delta}$$

$$\Pi_{\rho_y{}^n,\delta}$$

$$\Pi^n_{\rho,\delta}$$

Decoding POVM for Quantum MAC
(ctd.)

Define decoding POVM *in the commuting case* to be:

$$\Lambda_{l,m} \equiv \left(\sum_{l',m'} \Pi'_{l',m'} \right)^{-1/2} \Pi'_{l,m} \left(\sum_{l',m'} \Pi'_{l',m'} \right)^{-1/2}$$

where

$$\Pi'_{l,m} \equiv \Pi^n_\rho \Pi_{\rho_{x^n(l)}} \Pi_{\rho_{y^n(m)}} \Pi_{\rho_{x^n(l),y^n(m)}}$$

Note: The above operator is **positive** if all $\rho_{x,y}$ commute.
Does not necessarily have to be otherwise.

MAC Error Analysis

Analyze **exp. of avg. error probability** for first message pair:

$$\mathbb{E}_{X^n,Y^n}\left[\mathrm{Tr}\left\{(I-\Lambda_{1,1})\rho_{x^n(1),y^n(1)}\right\}\right]$$

Exploit **Hayashi-Nagaoka operator inequality** to bound the error from above by **four terms**:

1) Probability that codeword is not typical:

$$\mathrm{Tr}\left\{\left(I-\Pi_{\rho_{x^n(1),y^n(1)},\delta}\right)\rho_{x^n(1),y^n(1)}\right\}\leq\epsilon$$

MAC Error Analysis (ctd.)

2) Probability that one message detected correctly but the other incorrectly:

$$\mathbb{E}_{X^n,Y^n}\left[\sum_{m'\neq 1}\mathrm{Tr}\left\{\Pi_{\rho_{x^n(1),y^n(m')}}\Pi_{\rho_{x^n(1)}}\rho_{x^n(1),y^n(1)}\Pi_{\rho_{x^n(1)}}\right\}\right] \leq |\mathcal{M}|\,2^{-nI(Y;BX)}$$

3) Symmetric case:

$$\mathbb{E}_{X^n,Y^n}\left[\sum_{l'\neq 1}\mathrm{Tr}\left\{\Pi_{\rho_{x^n(l'),y^n(1)}}\Pi_{\rho_{y^n(1)}}\rho_{x^n(1),y^n(1)}\Pi_{\rho_{y^n(1)}}\right\}\right] \leq |\mathcal{L}|\,2^{-nI(X;BY)}$$

4) Probability that both detected incorrectly:

$$\mathbb{E}_{X^n,Y^n}\left[\sum_{l',m'\neq 1}\mathrm{Tr}\left\{\Pi_{\rho_{x^n(l'),y^n(m')}}\Pi_{\rho}\rho_{x^n(1),y^n(1)}\Pi_{\rho}\right\}\right] \leq |\mathcal{L}|\,|\mathcal{M}|\,2^{-nI(XY;B)}$$

Quantum Simultaneous Decoding Conjecture

There exists a **doubly-indexed decoding POVM**

$$\{\Lambda_{l,m}\}$$

such that the following **error probability** is small:

$$\mathbb{E}_{X^n,Y^n}\left[\mathrm{Tr}\left\{(I-\Lambda_{1,1})\rho_{x^n(1),y^n(1)}\right\}\right]$$

regardless of whether $\rho_{x,y}$s commute

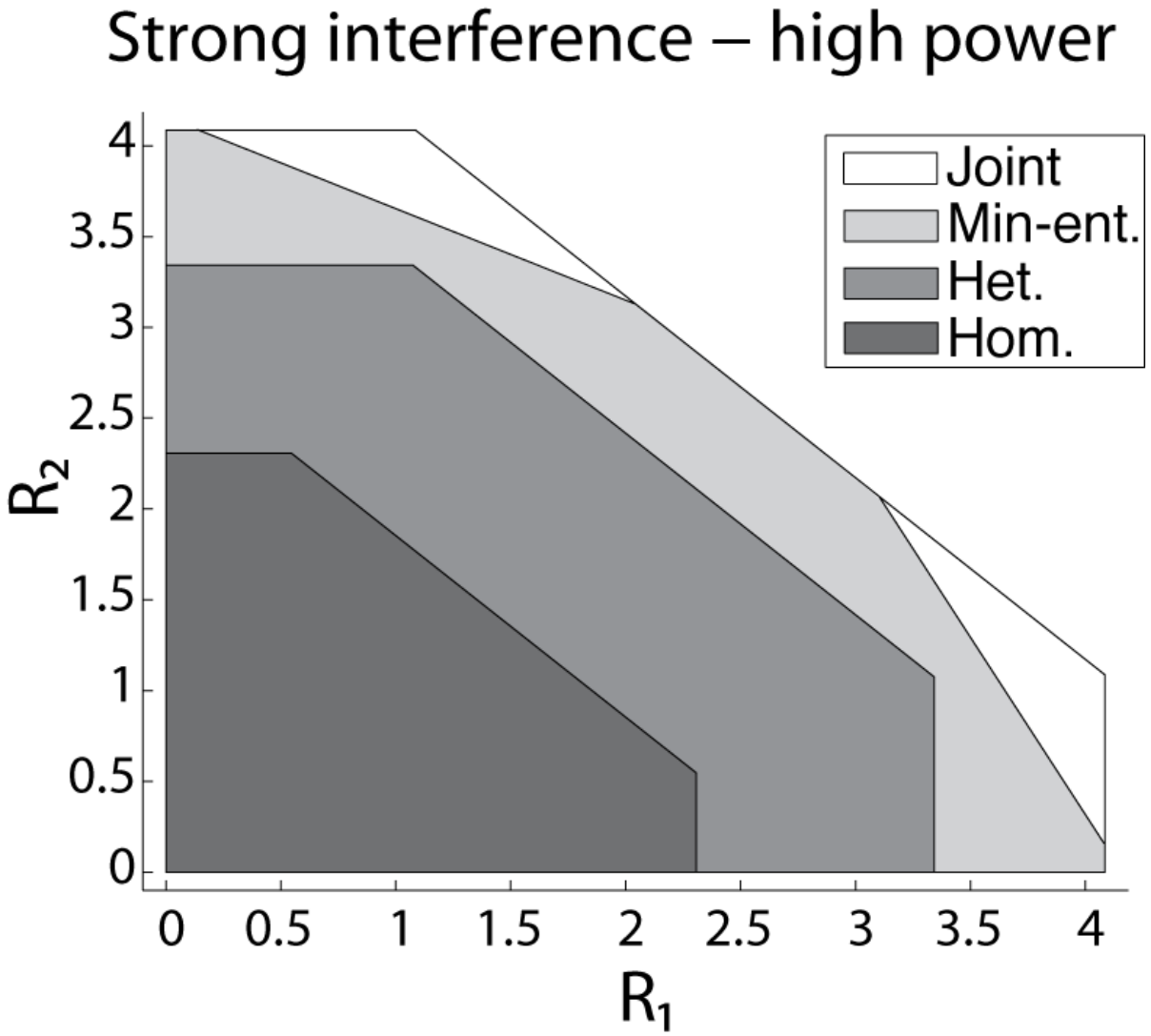
Tried to prove this in *many* ways (see arXiv:1102.2624)

1) simple way with **Gentle Operator Lemma** accumulates errors

2) **smooth min-entropies** (à la Renner)

3) **Feinstein approach** (à la Lloyd *et al*.)

Notable Success in Bosonic Comm.



A min-entropy decoder can outperform classical strategies in the context of comm. over a **bosonic interference channel**

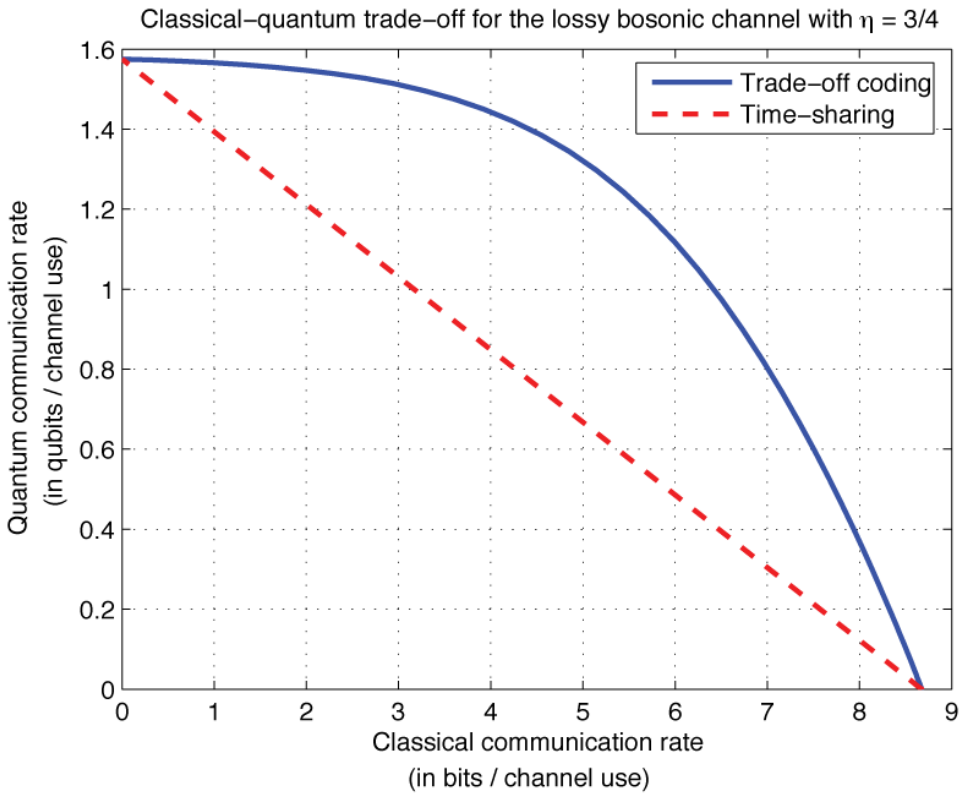
Guha, Savov, Wilde. arXiv:1102.2627 (accepted for ISIT in Russia)

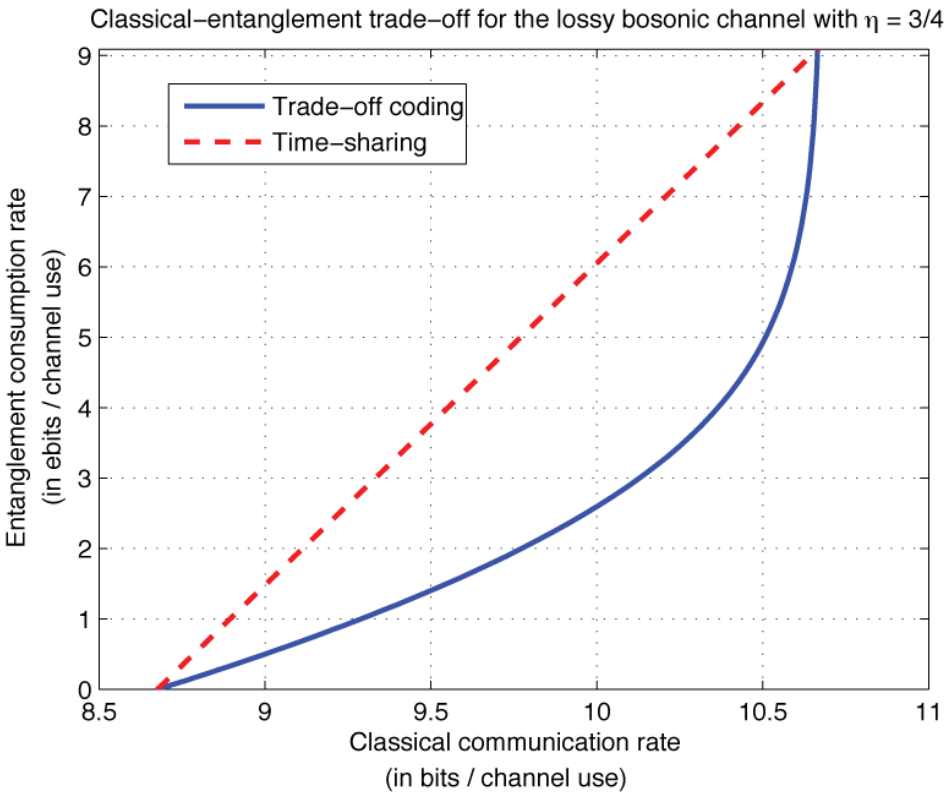
Conclusion

Solving this conjecture would allow us to import many results from **network information theory** for use in **quantum information theory**

For a **related conjecture** in multiparty state transfer, see the *PhD thesis* of *Nicolas Dutil* or **arXiv:1011.1974**

Advertisement of New Results





Trade-off coding for a **lossy bosonic channel** gives a remarkable gain over **time-sharing**

Wilde, Hayden, Guha. arXiv:1105.0119