Sequential, successive, and simultaneous decoders for entanglement-assisted classical communication (arXiv:1107.1347)

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Packing Argument for Sequential Decoding

A generalization of Giovannetti *et al.* [1]

$$\rho_m \xrightarrow{\Pi\Pi_1?} \rho_m \xrightarrow{\Pi\Pi_2?} \cdots \xrightarrow{\Pi\Pi_{m-1}?} \rho_m \xrightarrow{\Pi\Pi_m?} \text{YES!}$$

Given an ensemble $\{p_X(x), \rho_x\}_{x\in\mathcal{X}}$ where $\rho \equiv \sum_{x\in\mathcal{X}} p_X(x)\,\rho_x$. If there exists projectors Π and $\{\Pi_x\}_{x\in\mathcal{X}}$ such that:

$$\mathrm{Tr}\{\Pi\rho_x\} \geqslant 1-\epsilon, \qquad \Pi_x\rho_x\Pi_x \geqslant \frac{1}{d}\Pi_x,$$

$$\mathrm{Tr}\{\Pi_x\rho_x\} \geqslant 1-\epsilon, \qquad \Pi\rho\Pi \leqslant \frac{1}{D}\Pi,$$

$$[\Pi_x, \rho_x] = 0,$$

then we can encode messages from a set $\mathcal{M}$ with $|\mathcal{M}| \ll D/d$ using this ensemble and sequential decoding gives us that:

$$E\{p_{succ}\} \geqslant \left|(1-2\epsilon)\left(2 - e^{\frac{d}{D}|\mathcal{M}|}\right)\right|^2$$

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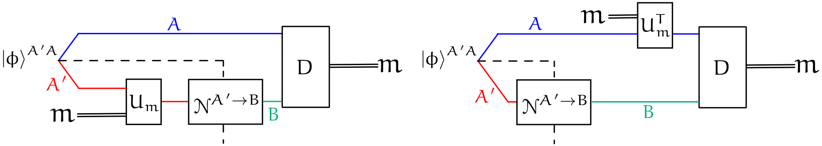
[1] Vittorio Giovannetti, Seth Lloyd, and Lorenzo Maccone. Achieving the Holevo bound via sequential measurements. December 2010. arXiv:1012.0386.

Sequential Decoding for EA Classical Communication

Many copies of the shared entanglement. For each message $m \in \mathcal{M}$, we randomly choose an encoder as in Hsieh *et al.* [2]:

$$|\phi\rangle^{A'A} = \sum_{t} \sqrt{p(t)}\, |\Phi_t\rangle^{A'A}, \quad U_m \equiv \bigoplus_{t} (-1)^{b_t} X(x_t)\, Z(z_t)\,.$$

Using the transpose trick:



Decoding projectors:

$$\Pi \equiv \Pi_{\delta}^{A} \otimes \Pi_{\delta}^{B}, \qquad \Pi_m \equiv U_m^{T} \Pi_{\delta}^{AB} U_m^{*}.$$

[2] Min-Hsiu Hsieh, Igor Devetak, and Andreas Winter. Entanglement-assisted capacity of quantum multiple-access channels. IEEE Transactions on Information Theory, 54(7):3078-3090, 2008.

Sequential Decoding for EA Classical Communication over a MAC

$$\rho_{l,m} \xrightarrow{\Pi\Pi_1?} \rho_{l,m} \xrightarrow{\Pi\Pi_2?} \cdots \xrightarrow{\Pi\Pi_{l-1}?} \rho_{l,m} \xrightarrow{\Pi\Pi_l?} \text{YES!}$$

$$\rho_{l,m} \xrightarrow{\Pi_l\Pi_{l,1}?} \rho_{l,m} \xrightarrow{\Pi_l\Pi_{l,2}?} \cdots \xrightarrow{\Pi_l\Pi_{l,m-1}?} \rho_{l,m} \xrightarrow{\Pi_l\Pi_{l,m}?} \text{YES!}$$

Given is $\{p_X(x)p_Y(y), \rho_{x,y}\}$, with some new requirements for the MAC setting:

$$\Pi_x\rho_x\Pi_x \geqslant \frac{1}{d_1^-}\Pi_x, \qquad \Pi_{x,y}\rho_{x,y}\Pi_{x,y} \geqslant \frac{1}{d_2}\Pi_{x,y},$$

$$\Pi\rho\Pi \leqslant \frac{1}{D_1}\Pi, \qquad \Pi_x\rho_x\Pi_x \leqslant \frac{1}{d_1^+}\Pi_{x,y},$$

$$[\Pi_x, \rho_x] = 0, \qquad [\Pi_{x,y}\rho_{x,y}] = 0.$$

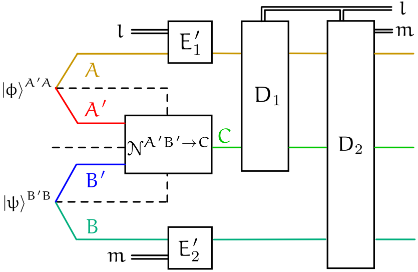
We then can encode message sets with size $|\mathcal{L}| \ll D_1/d_1^-$ and $|\mathcal{M}| \ll d_1^+/d_2$.

Simultaneous Decoding for EA Classical Communication

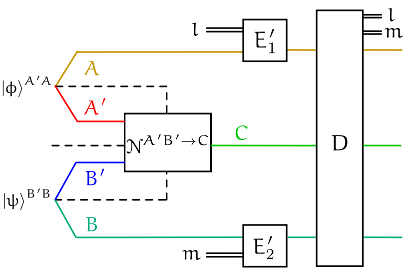
Two message sets $\mathcal{L}$ and $\mathcal{M}$. Shared entanglement sent through a noisy MAC. Same encoding unitary as above with the transpose trick:

$$\rho^{ABC} \equiv \mathcal{N}^{A'B' \to C} \left(|\phi\rangle \langle\phi|^{A'A} \otimes |\psi\rangle \langle\psi|^{B'B} \right),$$

$$\sigma_{lm} \equiv \left(U_l^T \otimes U_m^T\right) \rho^{ABC} \left(U_l^* \otimes U_m^*\right).$$



Successive decoding



Simultaneous decoding

Simultaneous Decoding for EA Classical Communication

This time we employ a *simultaneous* decoder:

$$\Lambda_{lm} \equiv \left(\sum_{l',m'} \Upsilon_{l'm'} \right)^{-\frac{1}{2}} \Upsilon_{lm} \left(\sum_{l',m'} \Upsilon_{l'm'} \right)^{-\frac{1}{2}},$$

$$\Upsilon \equiv U_l^T \hat{\Pi}_3 \hat{\Pi}_2 U_m^T \hat{\Pi} U_m^* \hat{\Pi}_2 \hat{\Pi}_3 U_l^*,$$

$$\hat{\Pi}_1 \equiv \Pi^A \otimes \Pi^{BC}, \qquad \hat{\Pi}_2 \equiv \Pi^B \otimes \Pi^{AC}, \qquad \hat{\Pi}_3 \equiv \Pi^C \otimes \Pi^{AB},$$

$$\hat{\Pi} \equiv \Pi^{ABC},$$

with the average error probability: $\frac{1}{|\mathcal{L}| \cdot |\mathcal{M}|} \sum_{l,m} \text{Tr}\{(I - \Lambda_{lm})\, \sigma_{lm}\},$

Simultaneous Decoding for EA Classical Communication

Instead of analyzing σ_{lm} directly, we "smooth" σ_{lm} into

$$\begin{aligned}\theta_{lm} &\equiv U_m^T \hat{\Pi}_1 U_m^* \sigma_{lm} U_m^* \hat{\Pi}_1 U_m^T \\ &= U_m^T \hat{\Pi}_1 U_l^T \rho^{ABC} U_l^* \hat{\Pi}_1 U_m^*,\end{aligned}$$

and use the following inequality

$$\text{Tr}\{\Gamma\rho\} \leqslant \text{Tr}\{\Gamma\sigma\} + \|\rho - \sigma\|_1 .$$

Using the Hayashi-Nagaoka inequality:

$$\bar{p}_e \leqslant \frac{1}{|\mathcal{L}| \cdot |\mathcal{M}|} \sum_{l,m} \left(2\text{Tr}\{(I - \Upsilon_{lm})\,\theta_{lm}\} + 4 \sum_{\substack{(l',m') \\ \neq (l,m)}} \text{Tr}\{\Upsilon_{l'm'}\theta_{lm}\} \right) + \epsilon.$$

Simultaneous Decoding in Quantum Communication

We can then run our protocol coherently à la Harrow [3]

$$\langle \mathcal{N} \rangle + H(A)\,[qq]_{AC} + H(B)\,[qq]_{BC} \geqslant R_1\,[q \to qq]_{AC} + R_2\,[q \to qq]_{BC}\,.$$

We then apply the coherent communication identity,

$$2\,[q \to qq] = [q \to q] + [qq]\,,$$

to obtain catalytic protocols for assisted and unassisted quantum communication.

[3] A. W. Harrow. Coherent communication of classical messages. Physical Review Letters, 92:097902, 2004.

EA Bosonic MAC

We give an achievable rate region for EA classical communication over a Bosonic MAC. We consider a beam-splitter

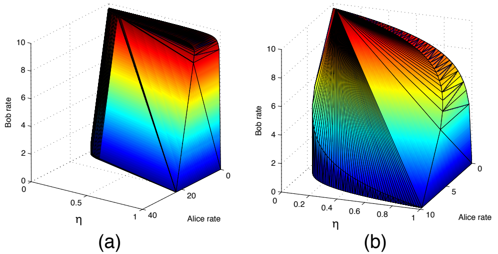
$$\hat{c} = \sqrt{\eta}\hat{a} + \sqrt{1-\eta}\hat{b},$$
$$\hat{e} = -\sqrt{1-\eta}\hat{a} + \sqrt{\eta}\hat{b},$$

with a two-mode squeezed vacuum as the input state:

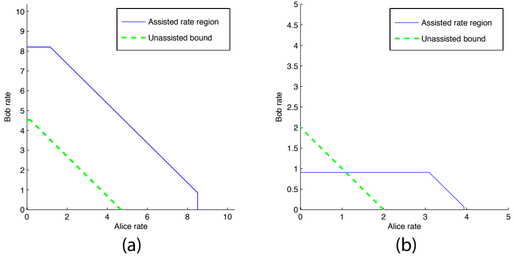
$$\sum_{n=0}^{\infty} \sqrt{\frac{N_S^n}{(N_S+1)^{n+1}}} \left|n\right\rangle \left|n\right\rangle .$$

Each sender shares a two-mode squeezed state with the receiver.

EA Bosonic MAC



Achievable rate region for $\eta \in [0, 1]$: (a) $N_{S_a} = 1000, N_{S_b} = 10$. (b) $N_{S_a} = 10, N_{S_b} = 10$.



Comparison with Yen-Shapiro outer bound: (a) $N_{S_a} = 10, N_{S_b} = 8, \eta = 1/2$. (b) $N_{S_a} = 1, N_{S_b} = 1, \eta = 0.95$.

Conclusions

In conclusion, we gave

- Extension to the sequential decoding and its applications.
- An EA simultaneous decoder and its applications.
- An achievable rate region for EA classical communication over a bosonic MAC and comparison with the unassisted case.

We should also mention that Sen [4] has recently improved the analysis for sequential decoding, and independently discovered an unassisted simultaneous decoder using sequential decoding.

[4] Pranab Sen. Achieving the Han-Kobayashi inner bound for the quantum interference channel by sequential decoding. September 2011. arXiv:1109.0802.