Information Trade-offs for Optical Quantum Communication

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arXiv:1105.0119

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McGill University, October 3, 2011

Overview

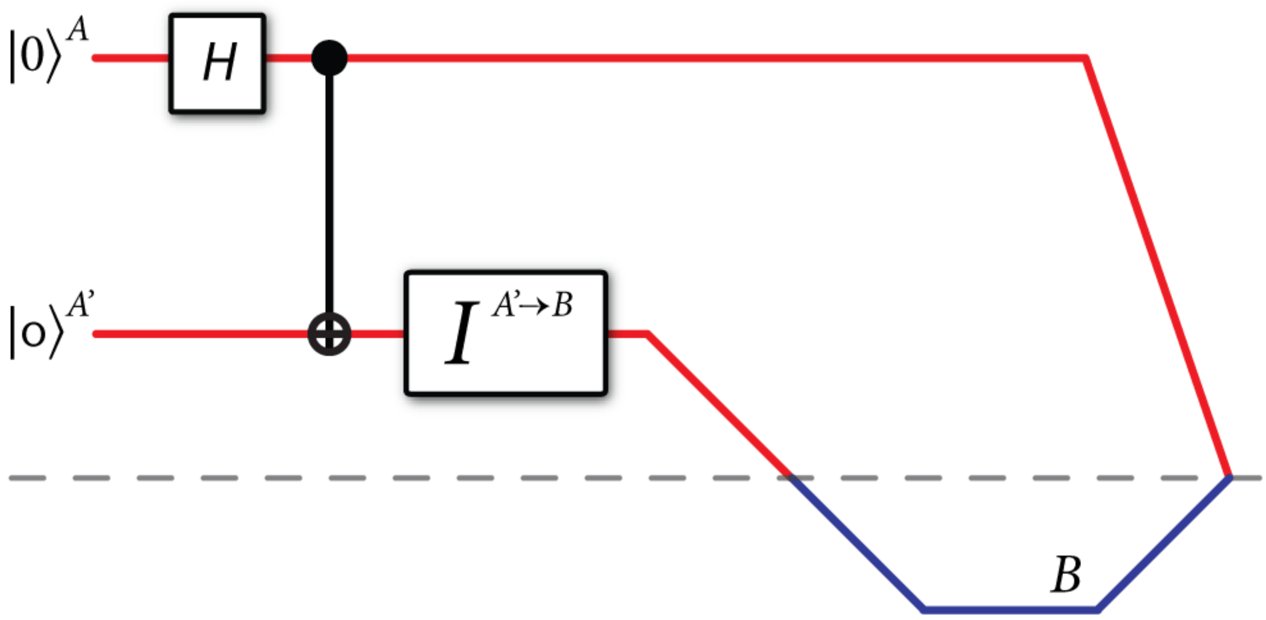
Review of basic protocols

Review of **classical capacity**, **quantum capacity**, and **entanglement-assisted capacity** of bosonic channels

The idea of encoding both **classical and quantum data**, etc.

Bosonic channels have *good trade-off curves*

Entanglement Distribution

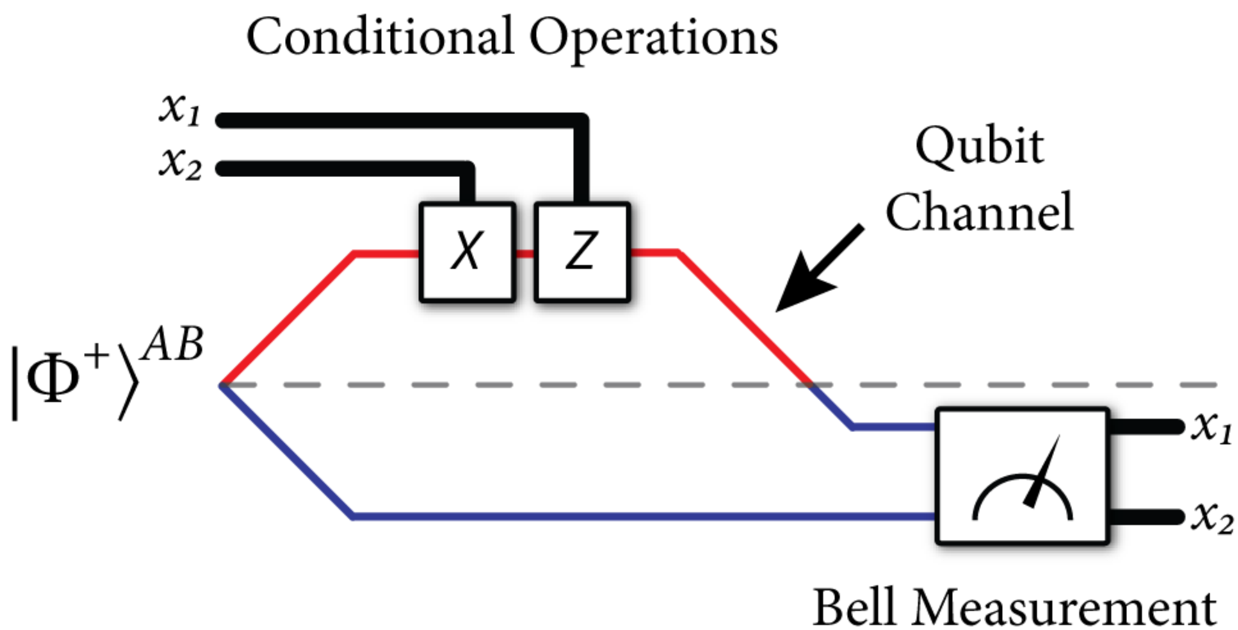


Trivial way to generate **entanglement** between **Alice** and **Bob**

$$|\Phi\rangle^{AB} \equiv \frac{1}{\sqrt{2}}\left(|00\rangle^{AB} + |11\rangle^{AB}\right)$$

(an *ebit*)

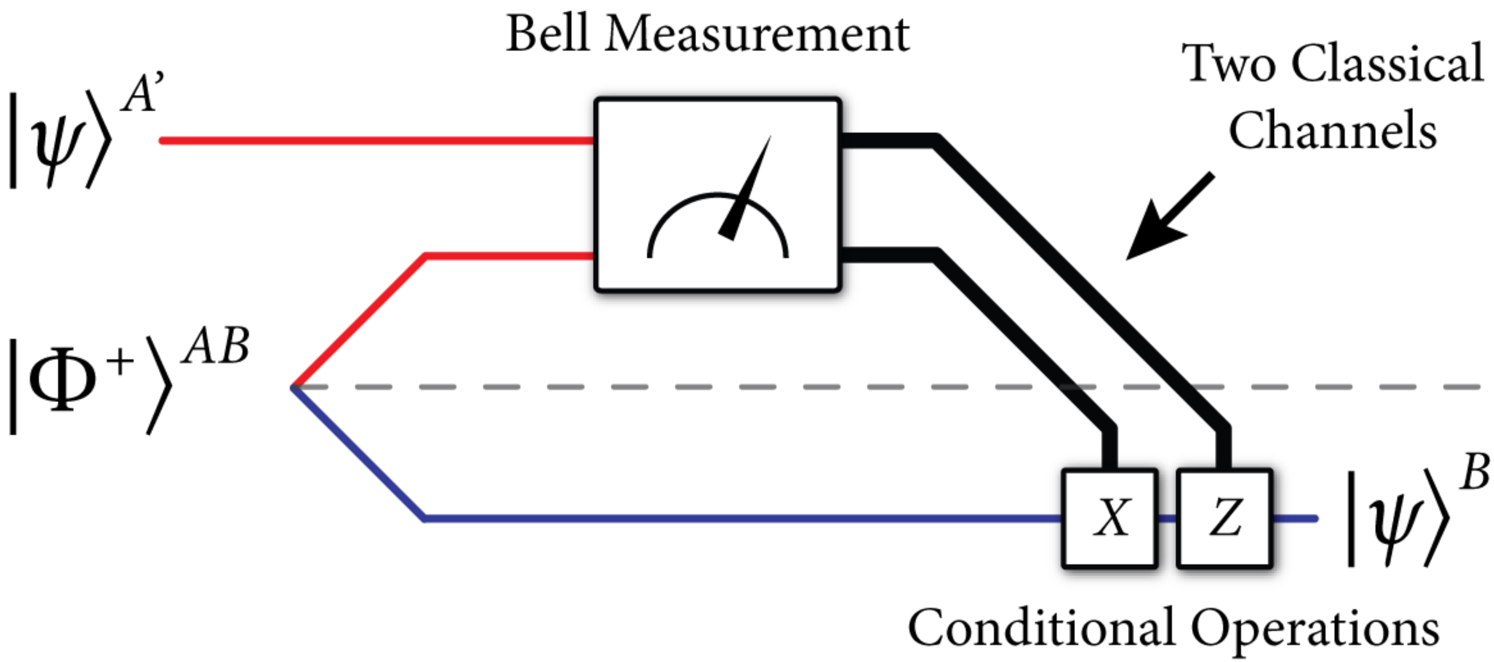
Super-dense Coding



One noiseless ebit and **one noiseless qubit channel** generates **two classical bit channels**

Bennett and Wiesner, *Physical Review Letters* 69, 2881 (1992)

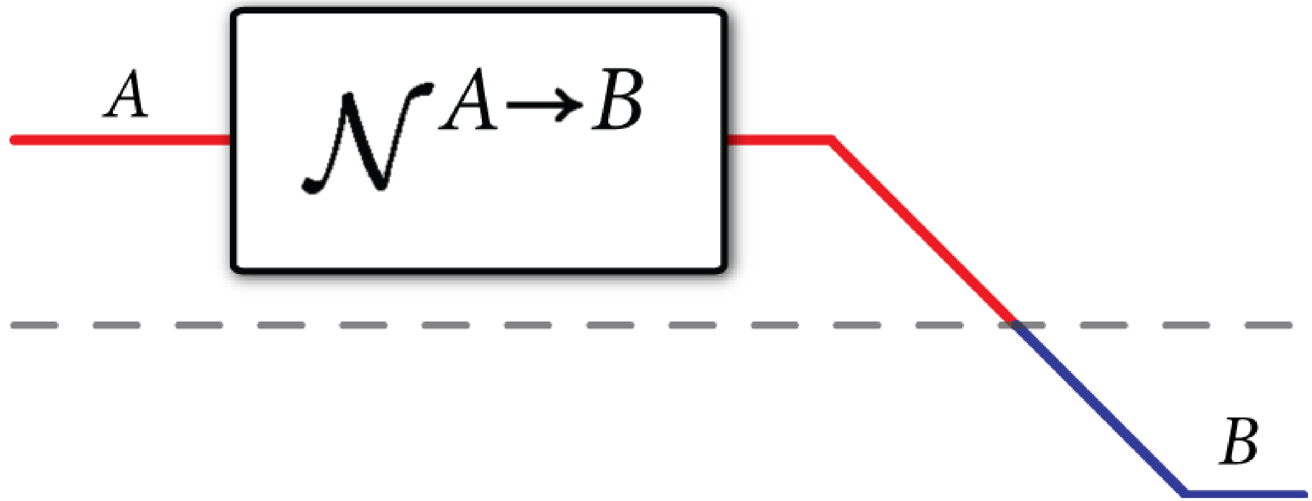
Teleportation



One noiseless ebit and **two classical bit channels** generates a **noiseless qubit channel** from Alice to Bob

Bennett *et al.*, *Physical Review Letters* 70, 1895 (1993)

Noisy Quantum Channel Model



Alice inputs a **density operator**: ρ_x^A

Bob gets **density operator**: ρ_x^B

Model channel as a
completely positive, trace-preserving map

How much information can Alice transmit to Bob (capacity)?

What is Capacity?

The ultimate **rate** at which two parties can communicate or perform some given task (*optimized over all possible encodings and decodings*).

Capacity theorem has two parts:

Direct Coding Theorem – For any rate below capacity, there exists a coding scheme that achieves that rate with vanishing error.

Converse Theorem – For any coding scheme with vanishing error, its rate is below capacity.

Sending Classical Information over a Quantum Channel

Coding Strategy

(similar to that for classical case)

Use the channel many times so that law of large numbers comes into play

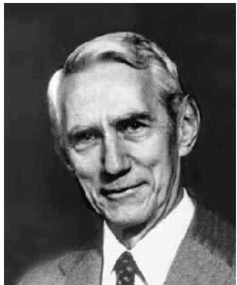
Code randomly with an ensemble of the following form:

$$\{p(x), \rho_x^{A'}\}_{x \in \mathcal{X}}$$

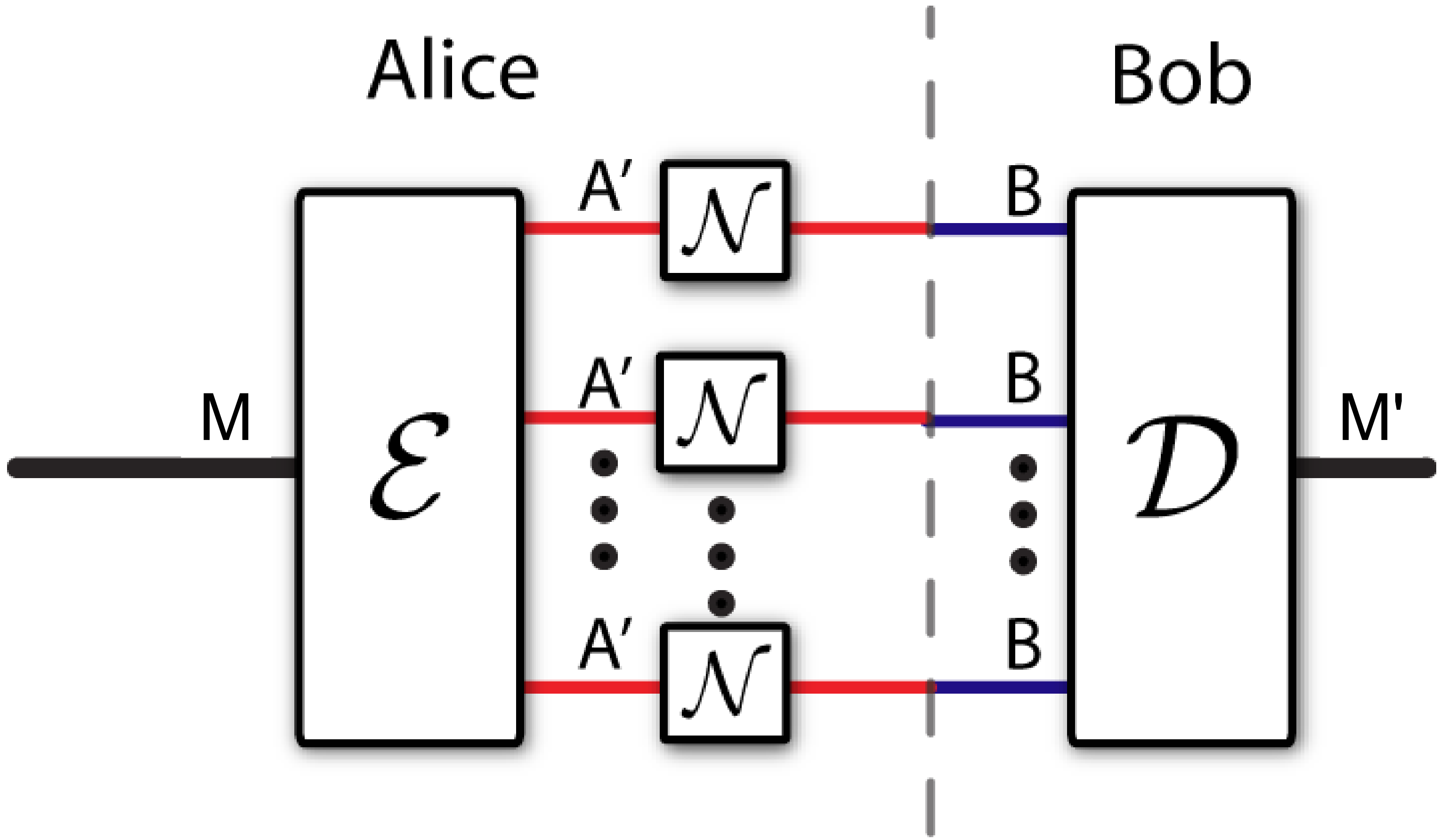
Channel input states are **product states**

Allow for small error but show that the error vanishe
large block length

Holevo, IEEE Trans. Inf. Theory, 44, 269-273 (1998).
Schumacher & Westmoreland, PRA, 56, 131-138 (1997).



Sending Classical Information over a Quantum Channel (ctd.)



Encoder just maps classical signal to a **tensor product state**

Decoder performs a measurement over all the output states to determine transmitted classical signal

Sending Classical Information over a Quantum Channel (ctd.)

Can achieve the following rate (bits/channel use):

$$I\left(X;B\right)_{\sigma} \quad \text{where} \quad \sigma^{XB} \equiv \sum_{x\in\mathcal{X}} p\left(x\right)\left|x\right\rangle\left\langle x\right|^{X} \otimes \mathcal{N}^{A'\to B}(\rho_x^{A'})$$

Holevo information of the channel:

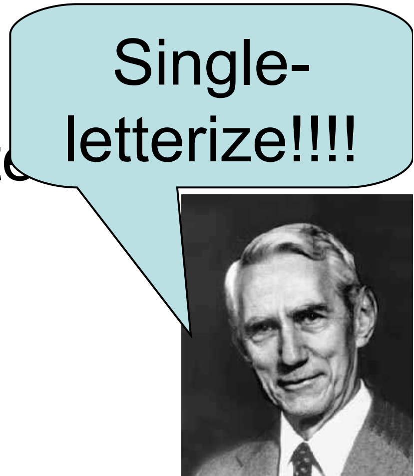
$$\chi(\mathcal{N}) \equiv \max_{\sigma} I\left(X;B\right)_{\sigma}$$

Capacity of the channel with product input states:

$$C(\mathcal{N}) = \chi(\mathcal{N})$$

Capacity of the channel with entangled input stat

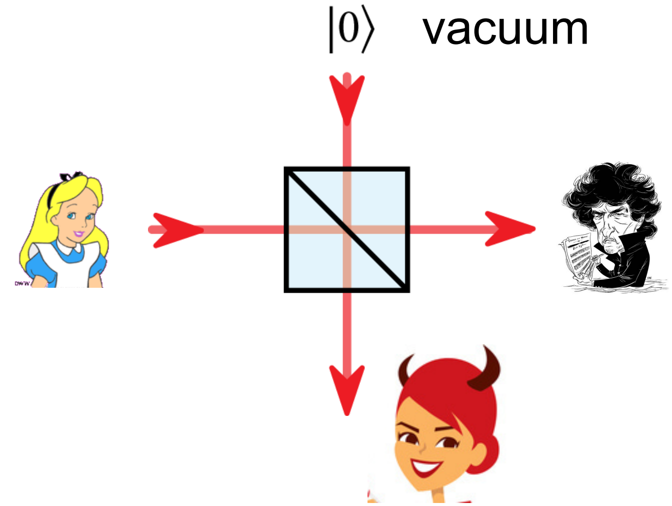
$$C(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n}\chi(\mathcal{N}^{\otimes n})$$



Bosonic Channels

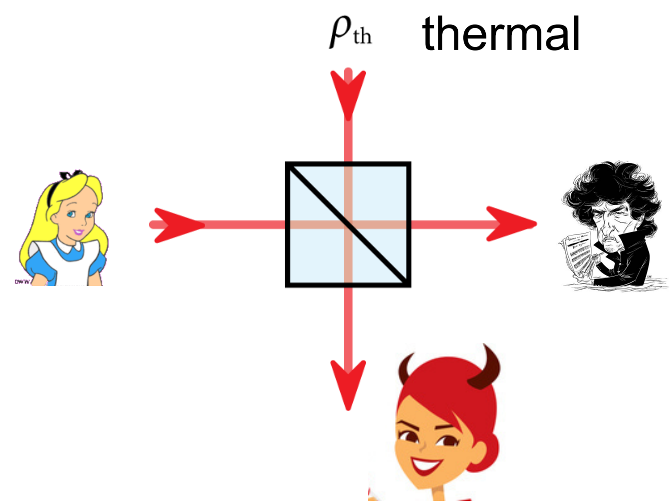
Lossy Bosonic Channel

(*models fiber optic or free space transmission*)



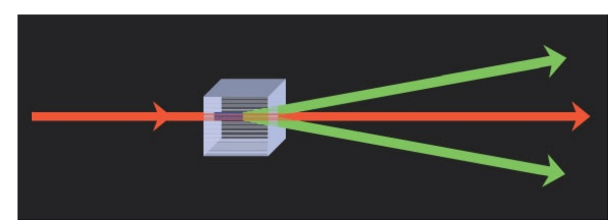
Thermalizing Channel

(similar model with background radiation)



Amplifier Channel

(models amplifier noise, Hawking-Unruh radiation)



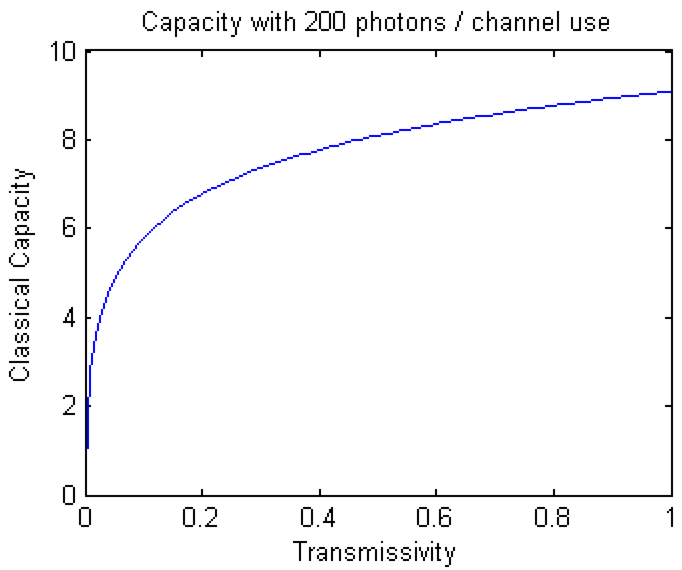
Weedbrook *et al*., Gaussian Quantum Information, *Reviews of Modern Physics* (2011).

Sending Classical Data over Bosonic Channels

Classical capacity of **lossy bosonic channel** is exactly

$$g(\eta N_S)$$

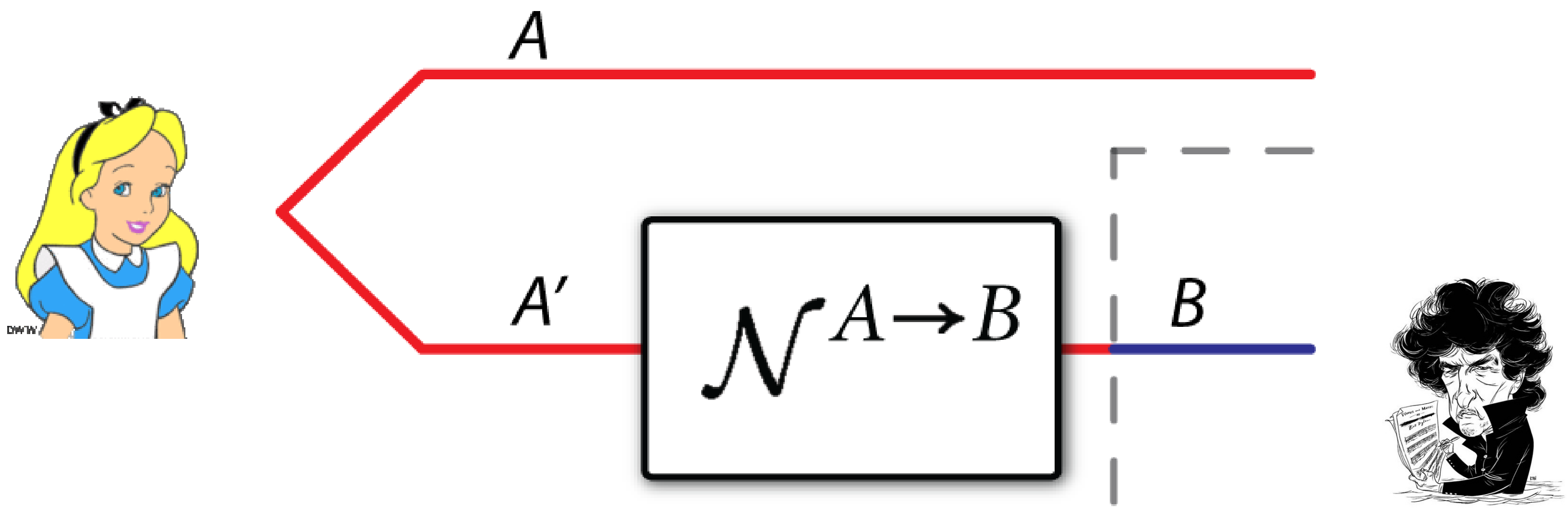
where η is **transmissivity** of channel, N_S is the **mean input photon number**, and $g(x) = (x+1)\log(x+1) - x \log x$ is the **entropy** of a **thermal state** with photon number x



Can **achieve** this capacity by selecting **coherent states** randomly according to a complex, isotropic Gaussian prior with variance N_S

Giovannetti *et al.*, *Physical Review Letters* 92, 027902 (2004)

Sending Quantum Data over Quantum Channels



Preserving entanglement is the same as transmitting quantum data

$$\mathcal{N}^{A'\to B}(\phi^{AA'})$$

Coherent information of a quantum channel:

$$Q(\mathcal{N}) \equiv \max_{\phi} I(A\rangle B)$$

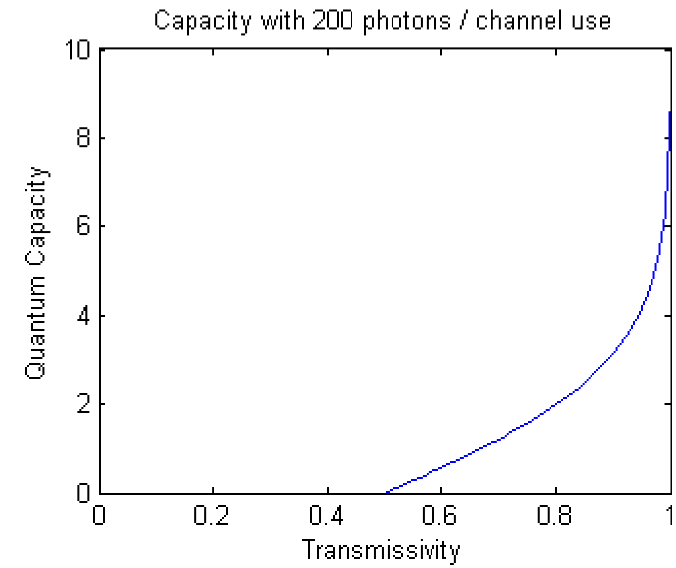
where $I(A\rangle B) \equiv H(B) - H(AB)$

Lloyd (1997), Shor (2002), Devetak (2005)

Sending Quantum Data over Bosonic Channels

Quantum capacity of lossy bosonic channel is

$$g(\eta N_S) - g((1-\eta)N_S)$$

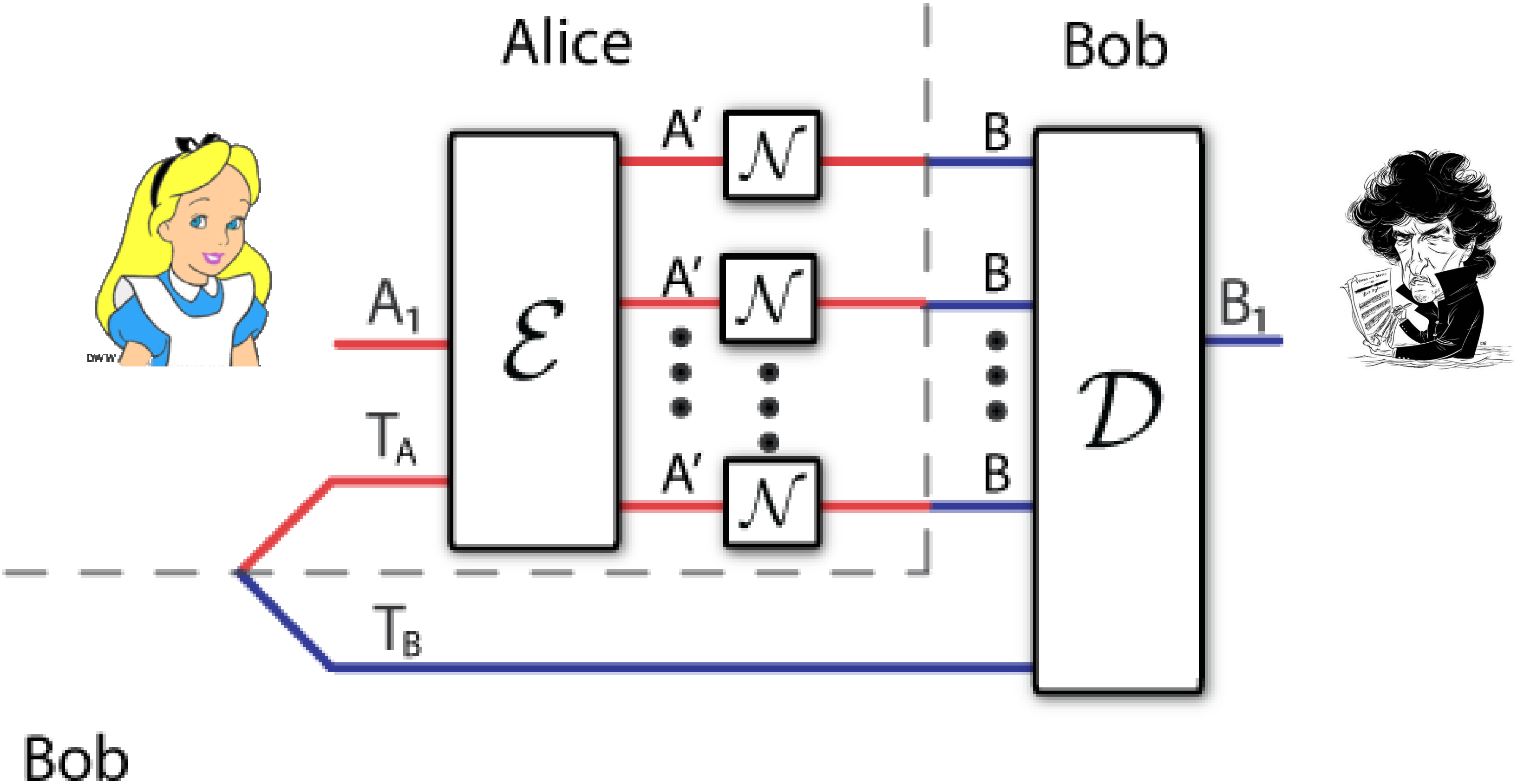


Interpretation: Generate **random** quantum codes from a **thermal state** distribution

An **achievable rate** is the *difference* of Bob and Eve's entropy

Holevo and Werner, *Physical Review A* 63, 032312 (2001)
Wolf *et al*., *Physical Review Letters* 98, 130501 (2007)
Guha *et al*., ISIT 2008, arXiv:0801.0841

Sending Quantum Data with Entanglement Assistance



Encoder is a random unitary mapping

Decoder decouples from Eve the quantum information Alice would like to protect

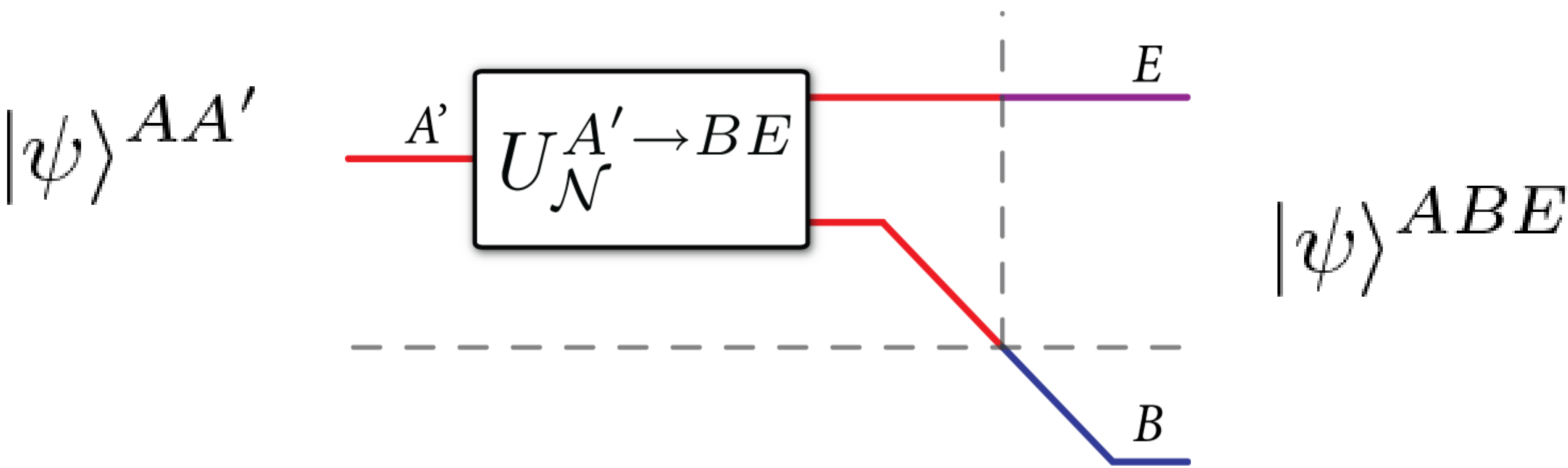
Devetak, Harrow, Winter, IEEE Trans. Information Theory vol. 54, no. 10, pp. 4587-4618, Oct 2008
Devetak, Harrow, Winter, Phys. Rev. Lett., 93, 230504 (2004).

Father Protocol

Can achieve the following resource inequality:

$$\langle \mathcal{N}^{A'\to B} \rangle + \frac{1}{2} I(A;E)_\psi [qq] \geq \frac{1}{2} I\left(A;B\right)_\psi \left[q \to q\right]$$

where



Devetak, Harrow, Winter, IEEE Trans. Information Theory vol. 54, no. 10, pp. 4587-4618, Oct 2008
Devetak, Harrow, Winter, Phys. Rev. Lett., 93, 230504 (2004).

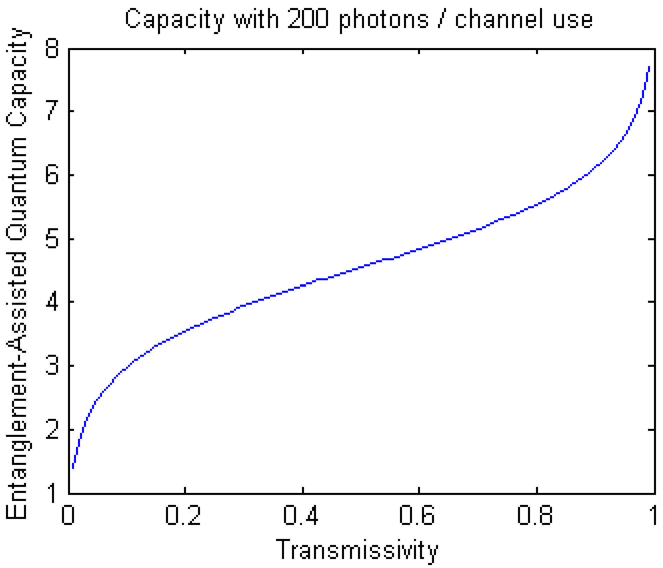
Entanglement-Assisted Quantum Transmission over Bosonic Channels

Entanglement-Assisted Quantum Capacity:

$$1/2\left[g(\eta N_S) + g(N_S) - g((1-\eta)N_S)\right]$$

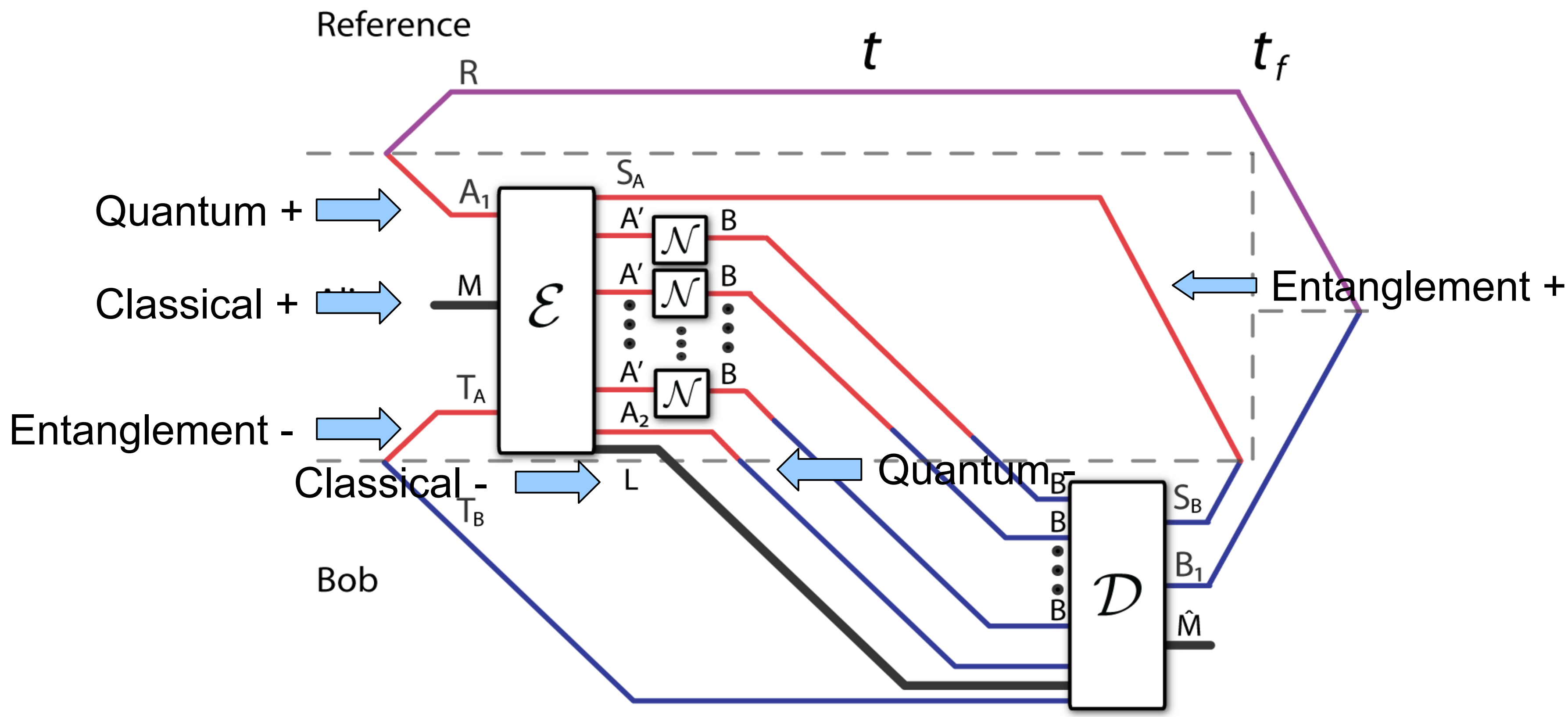
Again generate **random** quantum codes from a **thermal** distribution

Prior shared entanglement boosts capacity



Giovannetti et al., Physical Review Letters 91, 047901 (2003)
Wilde, Hayden, Guha. arXiv:1105.0119

The CQE Trade-off Setting



$$
\begin{aligned}
nC &= \log|M| - \log|L| \\
nQ &= \log|A_1| - \log|A_2| \\
nE &= \log|S_A| - \log|T_A|
\end{aligned}
$$

[1] Hsieh and Wilde. arXiv:0901.3038. *IEEE Transactions on Information Theory*, September 2010.
[2] Wilde and Hsieh. arXiv:1004.0458. The quantum dynamic capacity formula of a quantum channel.

Quantum Dynamic Capacity Theorem

The dynamic capacity region $\mathcal{C}_{CQE}(\mathcal{N})$ is

$$\mathcal{C}_{CQE}(\mathcal{N}) = \overline{\bigcup_{k=1}^{\infty} \frac{1}{k} \mathcal{C}_{CQE}^{(1)}(\mathcal{N}^{\otimes k})}. \tag{1}$$

The "one-shot" region $\mathcal{C}_{CQE}^{(1)}(\mathcal{N})$ is

$$\mathcal{C}_{CQE}^{(1)}(\mathcal{N}) \equiv \bigcup_{\sigma} \mathcal{C}_{CQE,\sigma}^{(1)}(\mathcal{N}).$$

The "one-shot, one-state" region $\mathcal{C}_{CQE,\sigma}^{(1)}(\mathcal{N})$ is the set of all rates C, Q, and E, such that

$$C + 2Q \leq I(AX;B)_{\sigma}, \tag{2}$$

$$Q + E \leq I(A\rangle BX)_{\sigma}, \tag{3}$$

$$C + Q + E \leq I(X;B)_{\sigma} + I(A\rangle BX)_{\sigma}. \tag{4}$$

The above entropic quantities are with respect to a classical-quantum state σ^{XAB} where

$$\sigma^{XAB} \equiv \sum_{x} p(x) \left|x\right\rangle \left\langle x\right|^{X} \otimes \mathcal{N}^{A'\to B}(\phi_x^{AA'}). \tag{5}$$

One should consider states on A'^k instead of A' when taking the regularization.

Achievability

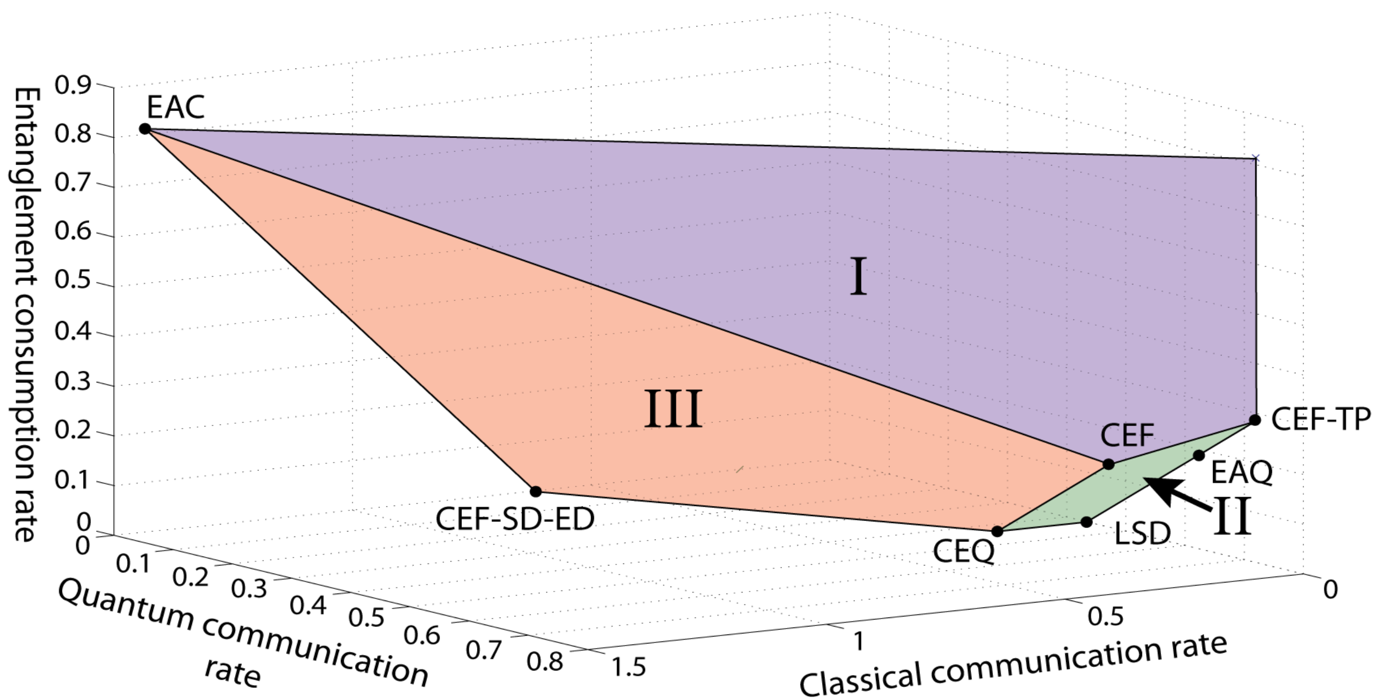
There exists a protocol for

entanglement-assisted classical and quantum communication

that achieves the following rates:

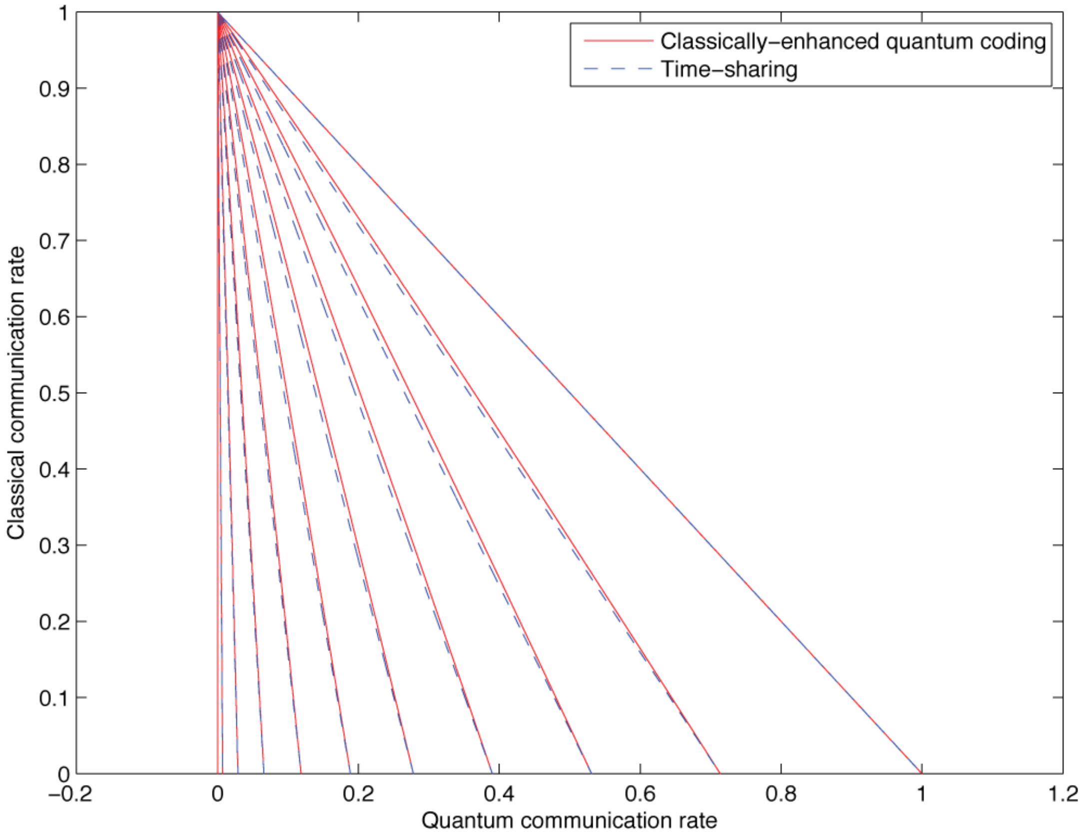
$$\langle \mathcal{N}^{A' \to B} \rangle + \frac{1}{2} I(A;E|X)_\sigma [qq] \geq \frac{1}{2} I(A;B|X)_\sigma [q \to q] + I(X;B)_\sigma [c \to c]$$

Combine this with teleportation, dense coding, and entanglement distribution...



Hsieh and Wilde. arXiv:0811.4227. *IEEE Transactions on Information Theory*, September 2010.

Trade-off Coding for Dephasing Channels



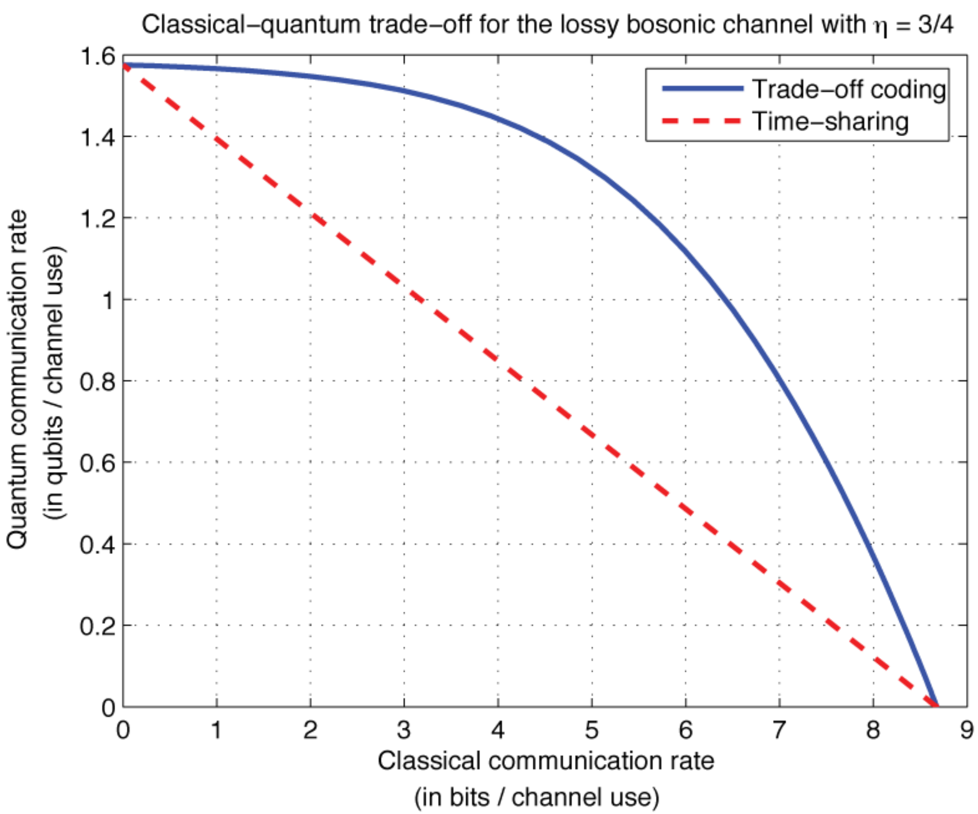
Classical-Quantum Trade-off

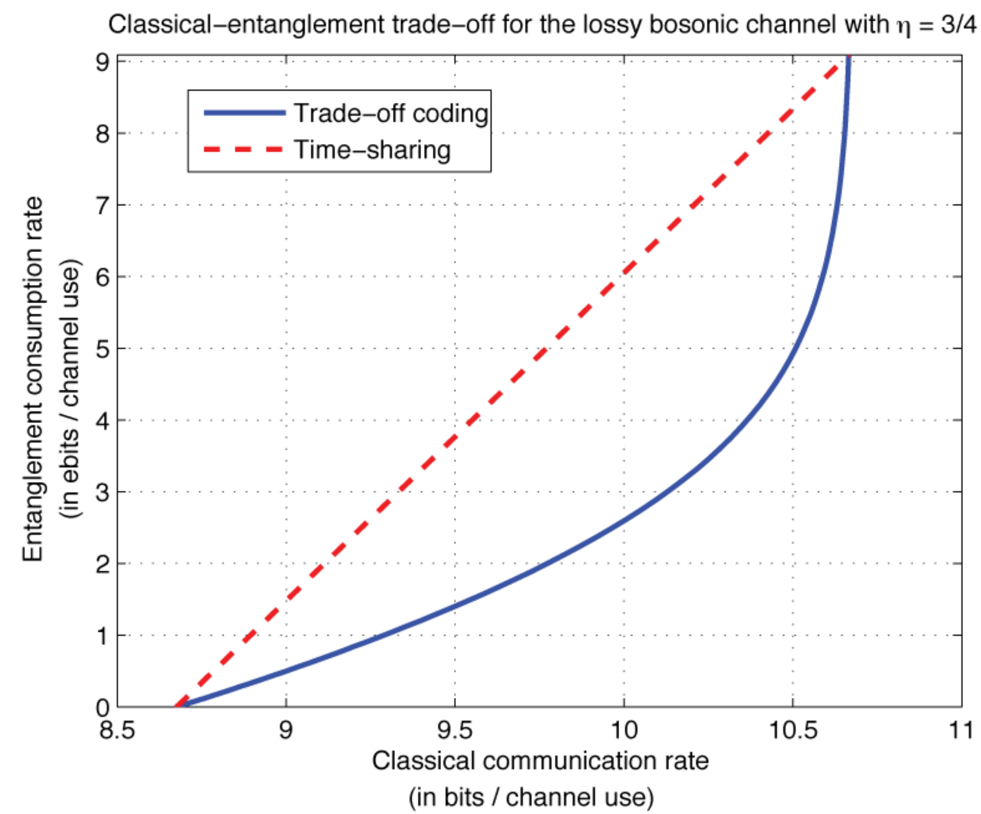
Trade-offs for a **qubit dephasing channel** with various noise levels *just barely* beat time-sharing

Why then would you implement a trade-off coding strategy in practice?

Bradler, Hayden, Touchette, Wilde. Physical Review A 81, 062312 (2010)

Trade-off Coding for Bosonic Channels





Classical-Quantum Trade-off

Classical-Ent. Trade-off

Trade-off is *so strong* for **bosonic channels** that it would be **silly** not to use such a strategy

Wilde, Hayden, Guha. arXiv:1105.0119

Power-Sharing Coding Strategy

Coding Ensemble:

$$\left\{p_{(1-\lambda)N_S}(\alpha), D^{A'}(\alpha)\left|\psi_{\mathrm{TMS}}\right\rangle^{AA'}\right\}.$$

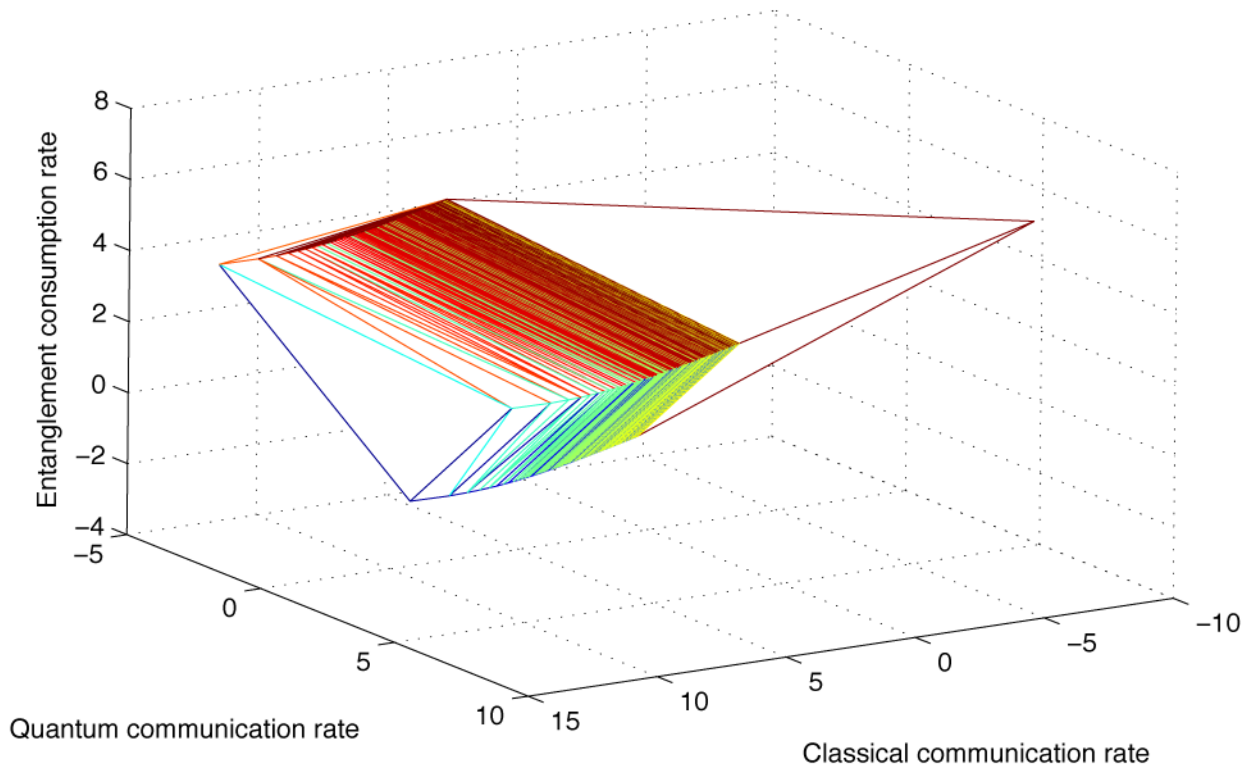
where

$$p_{(1-\lambda)N_S}(\alpha) \equiv \frac{1}{\pi(1-\lambda)N_S}\exp\left\{-\left|\alpha\right|^2/(1-\lambda)N_S\right\}$$

$$\left|\psi_{\mathrm{TMS}}\right\rangle^{AA'} \equiv \sum_{n=0}^{\infty}\sqrt{\frac{\left[\lambda N_S\right]^n}{\left[\lambda N_S+1\right]^{n+1}}}\left|n\right\rangle^A\left|n\right\rangle^{A'}$$

Wilde, Hayden, Guha. arXiv:1105.0119

Achievable Rate Region for Lossy Channel



$$C + 2Q \leq g(\lambda N_S) + g(\eta N_S) - g((1-\eta)\lambda N_S),$$

$$Q + E \leq g(\eta\lambda N_S) - g((1-\eta)\lambda N_S),$$

$$C + Q + E \leq g(\eta N_S) - g((1-\eta)\lambda N_S)$$

λ is a **power-sharing parameter** between zero and one

Wilde, Hayden, Guha. arXiv:1105.0119

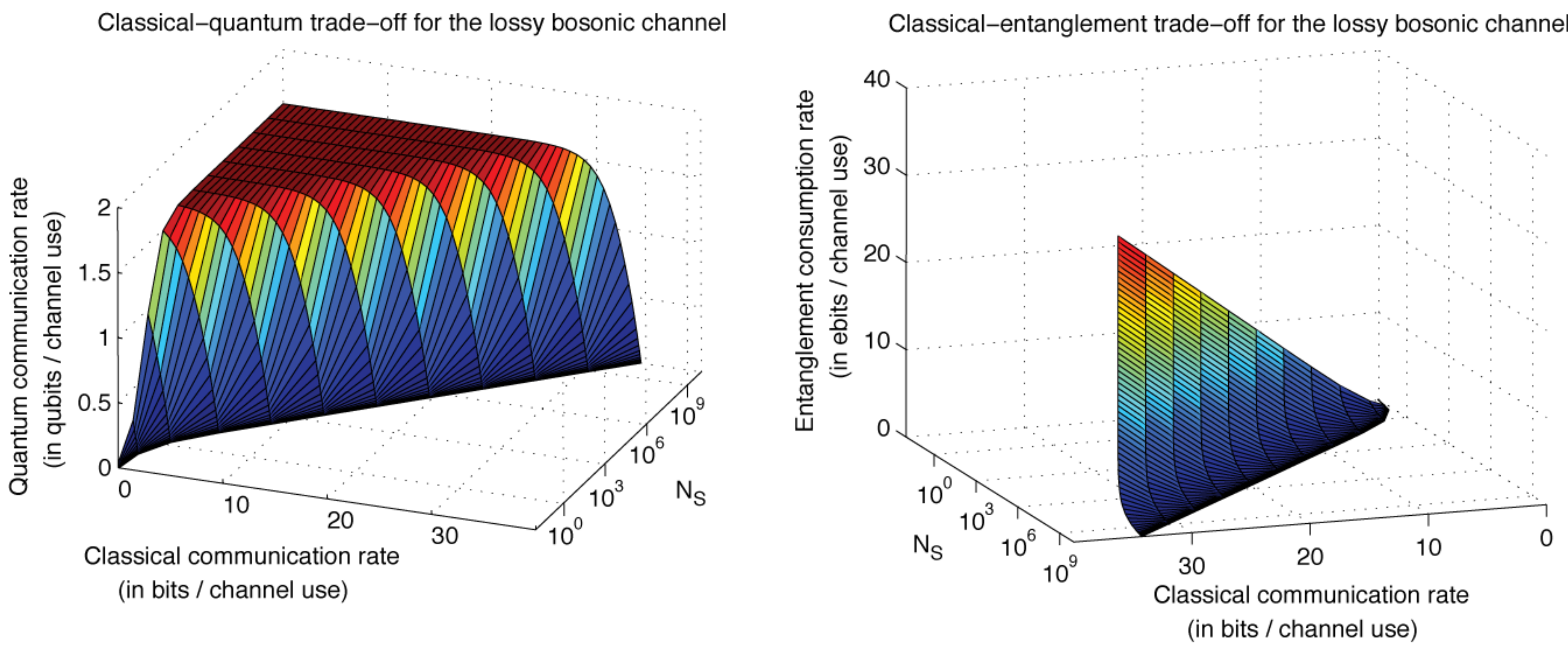
Rule of Thumb for Trade-off Coding

To be within ϵ bits of quantum capacity, choose

$$\lambda = 1/\left[\eta\left(1-\eta\right)\epsilon N_S \ln 2\right]$$

To be within ϵ bits of EA capacity, choose

$$\lambda = 5/\left[6\epsilon N_S \left(1-\eta\right)\ln 2\right]$$



Wilde, Hayden, Guha. arXiv:1105.0119

Conclusion

Power-sharing significantly outperforms **time-sharing** between the best known protocols

Do there exist structured encoders and decoders to achieve these rates?

Where to learn more about **quantum Shannon theory**:
Book available at arXiv:1106.1445