Entanglement Boosts Quantum Turbo Codes

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Joint work with **Min-Hsiu Hsieh**
arXiv:1010.????

Seminar for the Quantum Computing Group at McGill
Montreal, Quebec, Canada
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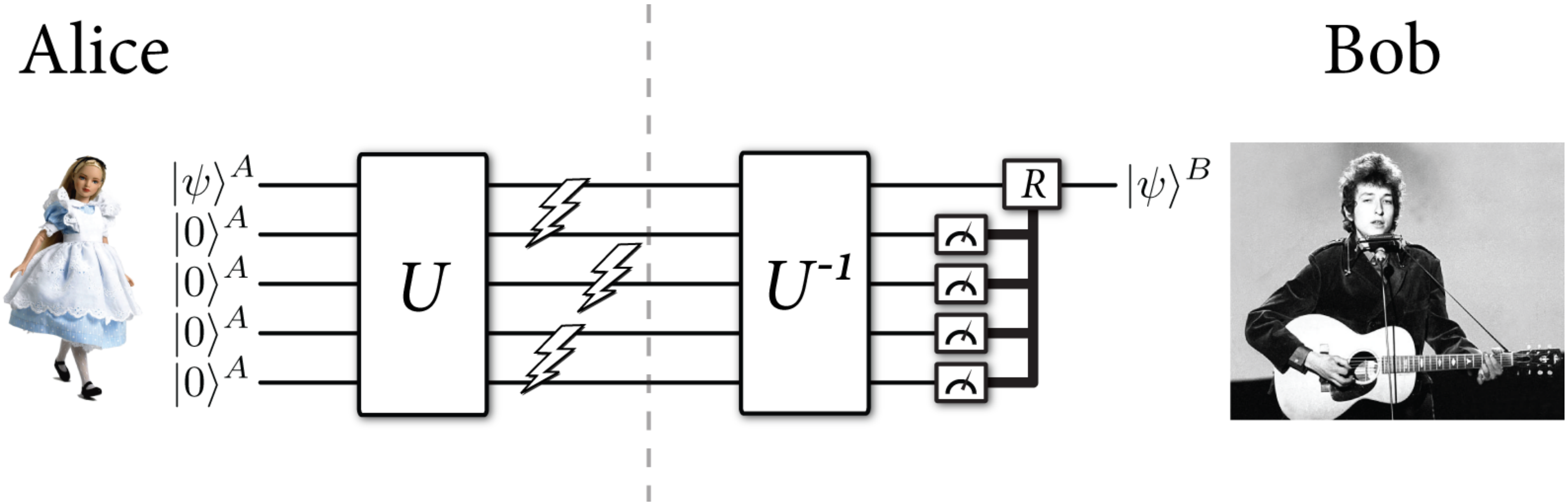
Overview

- Brief review of **quantum error correction** (entanglement-assisted as well)
- Review of **quantum convolutional codes** and their properties
- Adding **entanglement assistance** to quantum convolutional encoders
- Review of **quantum turbo codes** and their decoding algorithm
- Results of **simulating** entanglement-assisted turbo codes

Quantum Error Correction



*Shor, PRA **52**, pp. R2493-R2496 (1995).*

Stabilizer Formalism

Unencoded Stabilizer

Encoded Stabilizer

$$\begin{array}{ccccc} I & Z & I & I & I \\ I & I & Z & I & I \\ I & I & I & Z & I \\ I & I & I & I & Z \end{array} \longrightarrow \begin{array}{ccccc} X & Z & Z & X & I \\ I & X & Z & Z & X \\ X & I & X & Z & Z \\ Z & X & I & X & Z \end{array}$$

Unencoded Logical Operators

Encoded Logical Operators

$$\begin{array}{ccccc} X & I & I & I & I \\ Z & I & I & I & I \end{array} \longrightarrow \begin{array}{ccccc} X & X & X & X & X \\ Z & Z & Z & Z & Z \end{array}$$

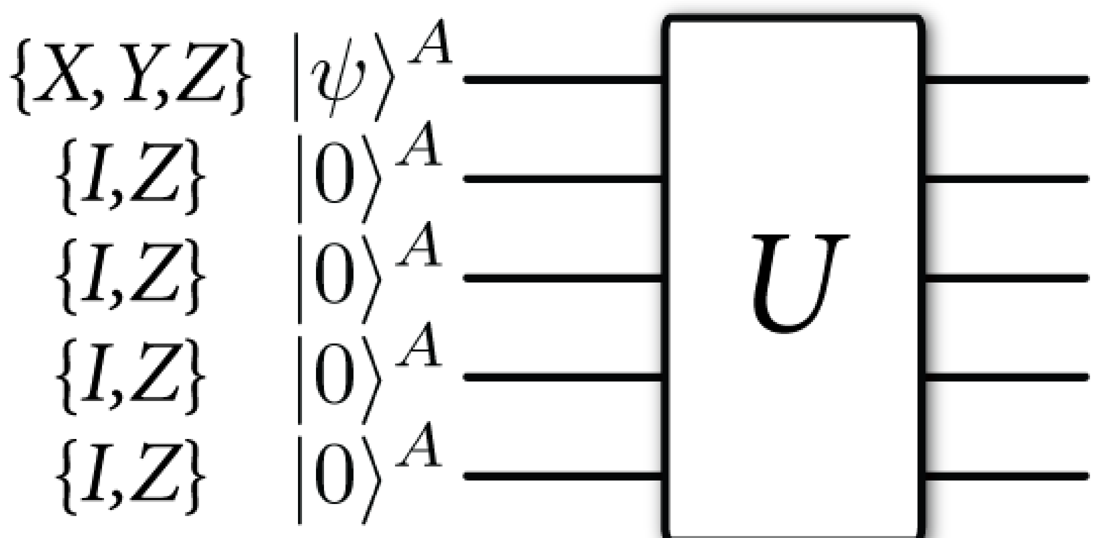
Laflamme et al., Physical Review Letters ***77****, 198-201 (1996).*

Distance of a Quantum Code

Distance is one indicator of a code's error correcting ability

It is the **minimum weight** of a logical operator that changes the encoded quantum information in the code

To determine distance, feed in X,Y, Z acting on logical qubits and I, Z acting on ancillas:

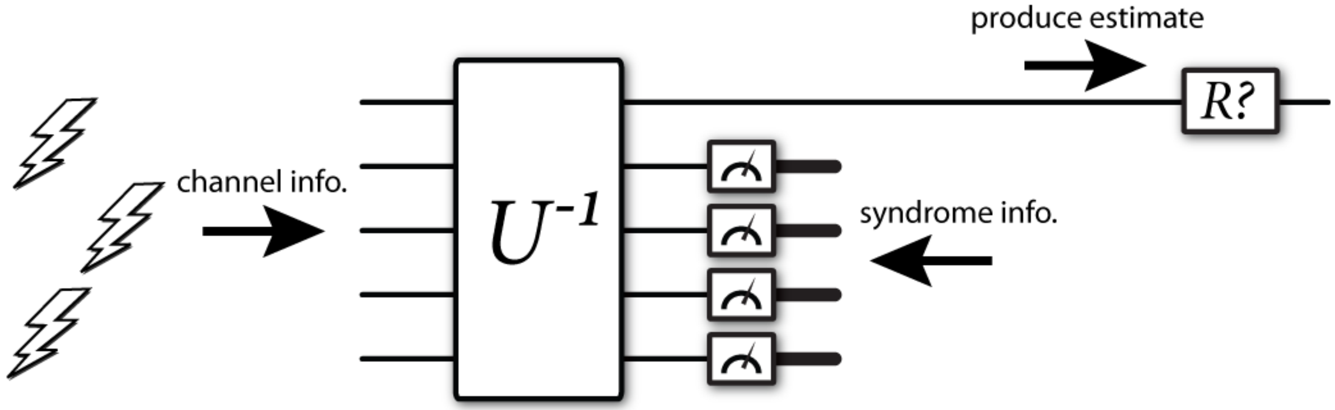


Distance is the minimum weight of all resulting operators

Maximum Likelihood Decoding

Find the most likely error
consistent with the channel model and the syndrome

Bob's job

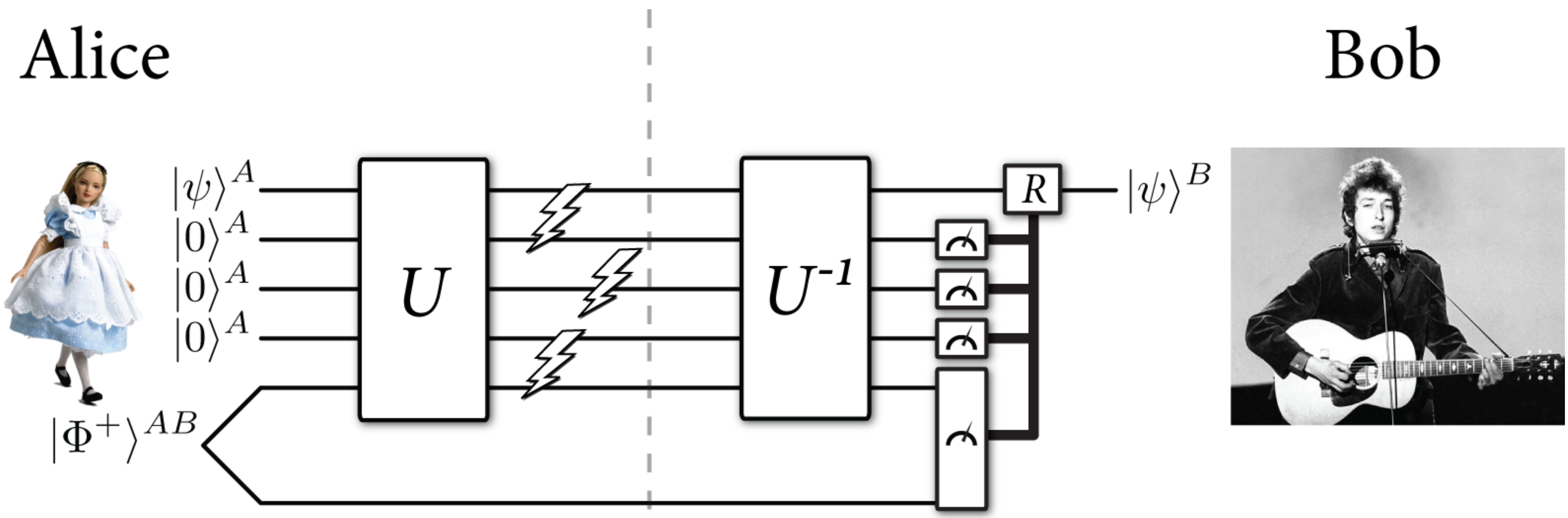


MLD decision is $\mathrm{argmax}_R \Pr\{R|S^x\}$

where

$$\Pr\{R, S^x\} = \sum_{S^z \in \{I,Z\}^{n-k}} \Pr\{P\}|_{P=(R:S^x+S^z)U}$$

Entanglement-Assisted Quantum Error Correction



Brun, Devetak, Hsieh. Science (2006)

Entanglement-Assisted Stabilizer Formalism

Unencoded Stabilizer

$$\left.\begin{array}{ccccc} I & Z & I & I & I \\ I & I & Z & I & I \\ I & I & I & Z & I \\ I & I & I & I & Z \\ I & I & I & I & X \end{array}\right| \begin{array}{c} I \\ I \\ I \\ Z \\ X \end{array} \longrightarrow \left.\begin{array}{ccccc} X & Z & Z & X & I \\ I & X & Z & Z & X \\ X & I & X & Z & Z \\ Z & X & I & X & Z \\ I & Z & I & I & Z \end{array}\right| \begin{array}{c} I \\ I \\ I \\ Z \\ X \end{array}$$

Encoded Stabilizer

Unencoded Logical Operators

$$\left.\begin{array}{ccccc} X & I & I & I & I \\ Z & I & I & I & I \end{array}\right| \begin{array}{c} I \\ I \end{array} \longrightarrow \left.\begin{array}{ccccc} X & X & X & X & X \\ Z & Z & Z & Z & Z \end{array}\right| \begin{array}{c} I \\ I \end{array}$$

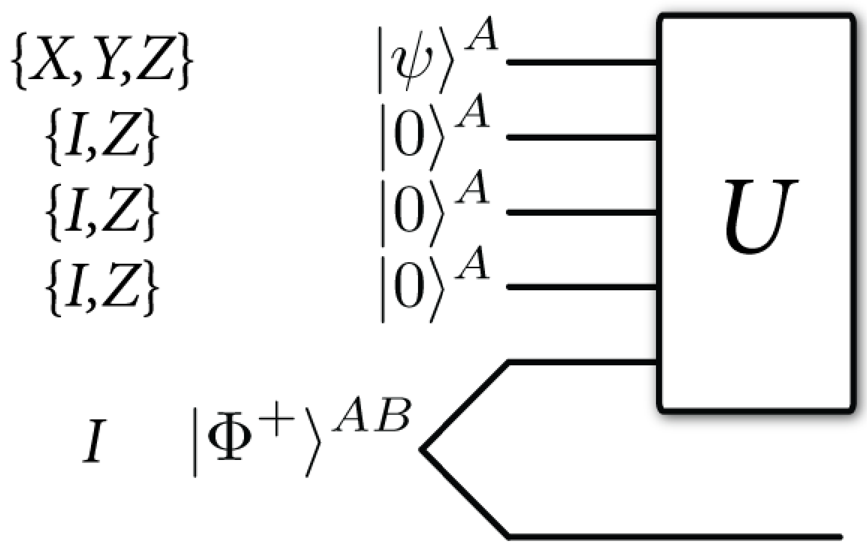
Encoded Logical Operators

Distance of an EA Quantum Code

Distance definition is nearly the same

It is the **minimum weight** of a logical operator that changes the encoded quantum information in the code

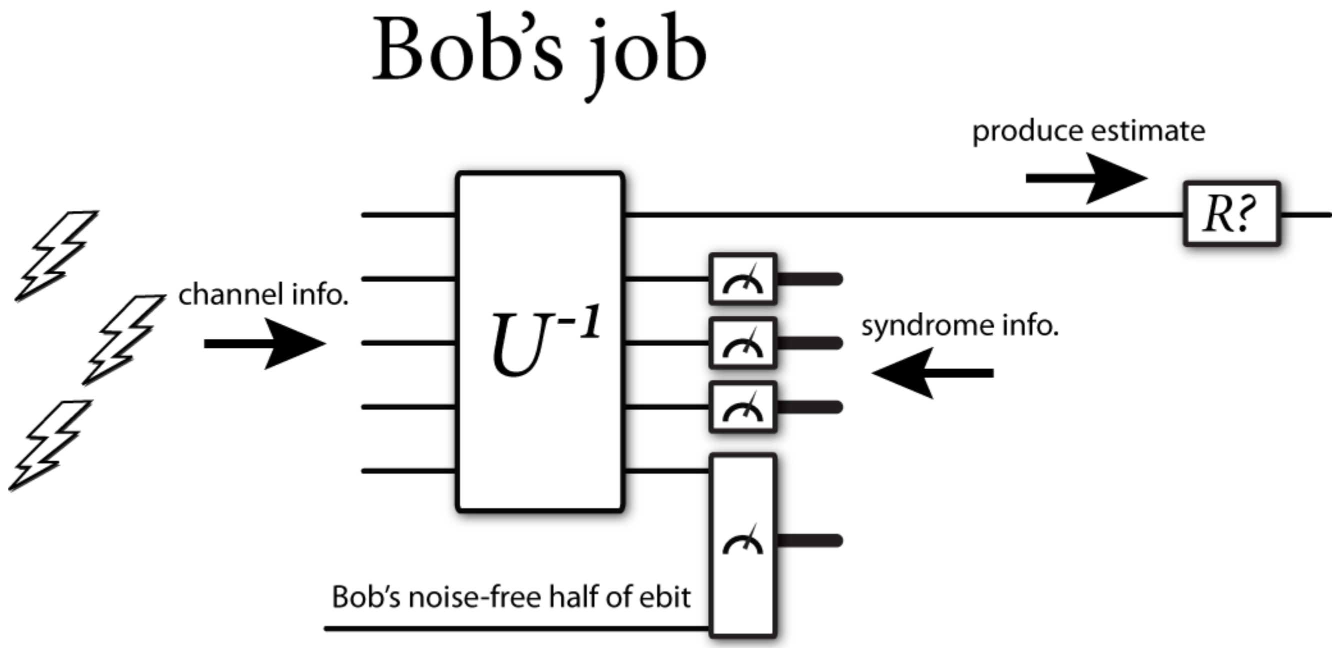
To determine distance, feed in *X*, *Y*, *Z* acting on logical qubits, *I*, *Z* acting on ancillas, and *I* acting on half of ebits:



Distance is the minimum weight of all resulting operators

EA Maximum Likelihood Decoding

Find the most likely error
consistent with the channel model and the syndrome

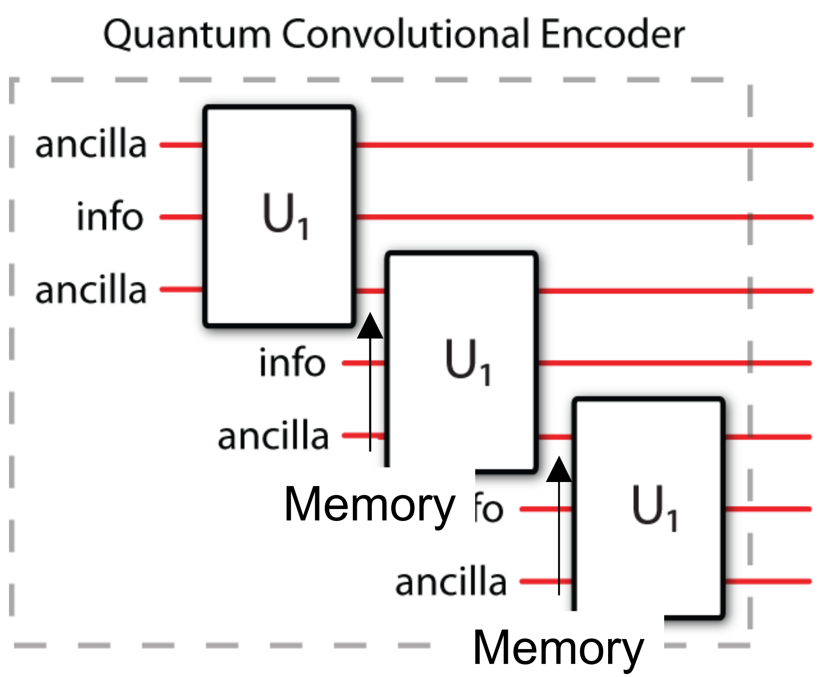


MLD decision is $\text{argmax}_R \Pr\{R|S^x, E^x, E^z\}$

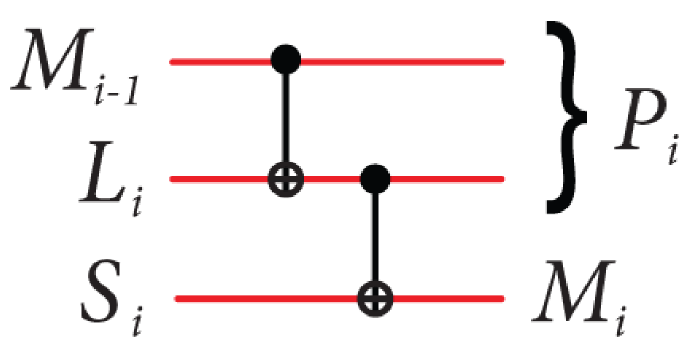
where

$$\Pr\{R, S^x, E^x, E^z\} = \sum_{S^z \in \{I,Z\}^{n-k}} \Pr\{P\}|_{P=(R:S^x+S^z:E^x+E^z)U}$$

Quantum Convolutional Codes



Example
:



H. Ollivier and J.-P. Tillich, “Description of a quantum convolutional code,” PRL (2003)

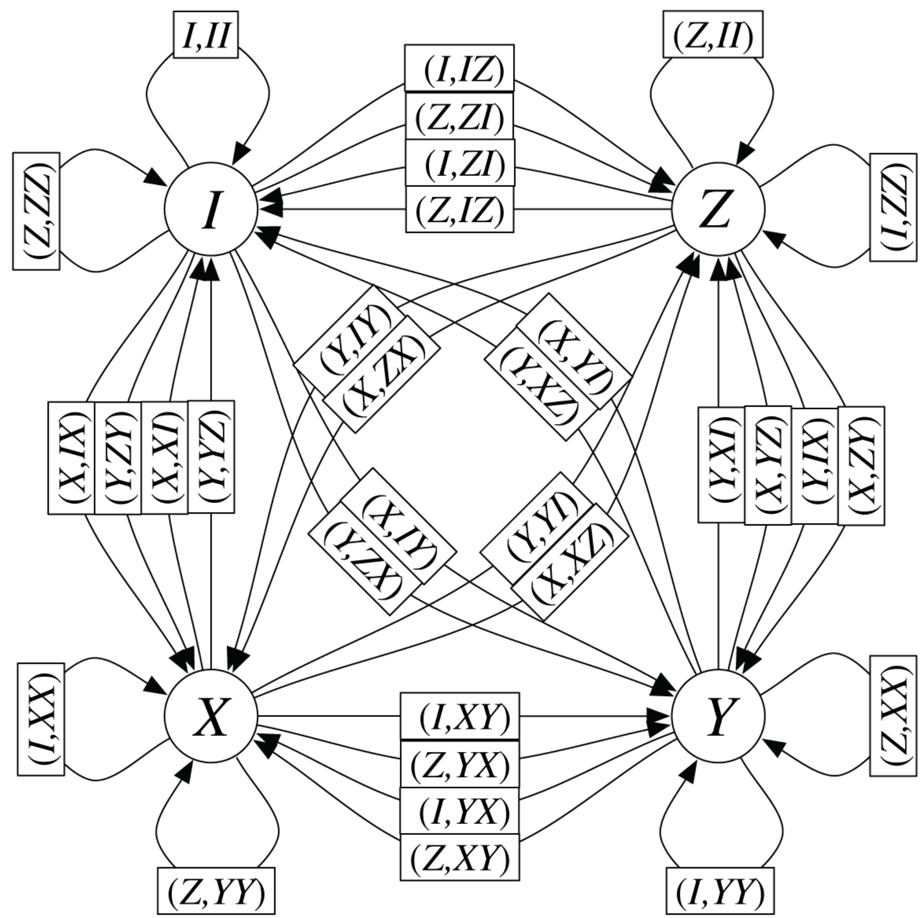
State Diagram

Useful for analyzing the properties of a quantum convolutional code

How to construct? Add an edge from one memory state to another if a logical operator and ancilla operator connects them:

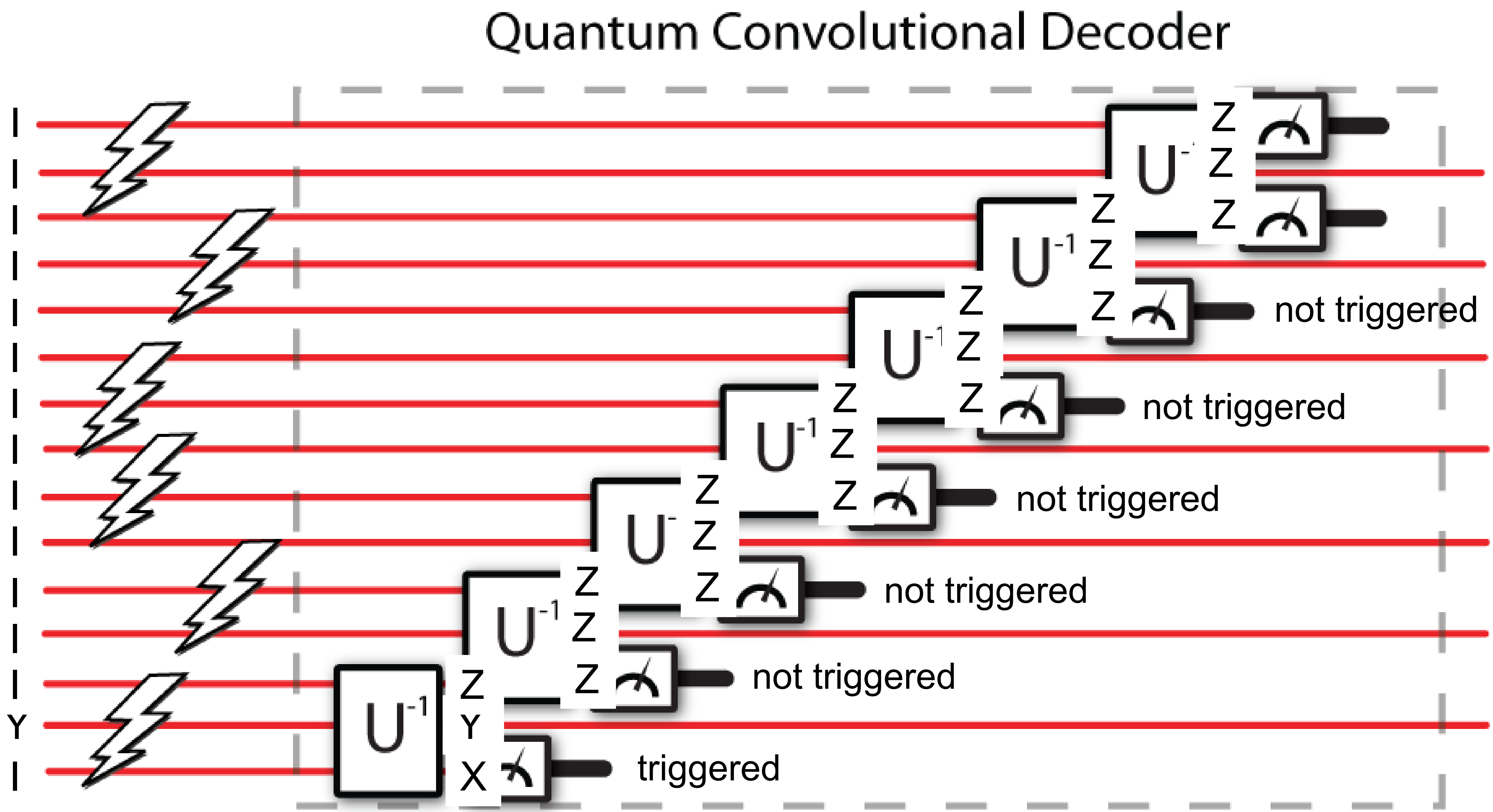
$$(M_{i-1} : L_i : S_i)U = (P_i : M_i)$$

State diagram for our example encoder



Tracks the flow of logical operators through the convolutional encoder

Catastrophicity

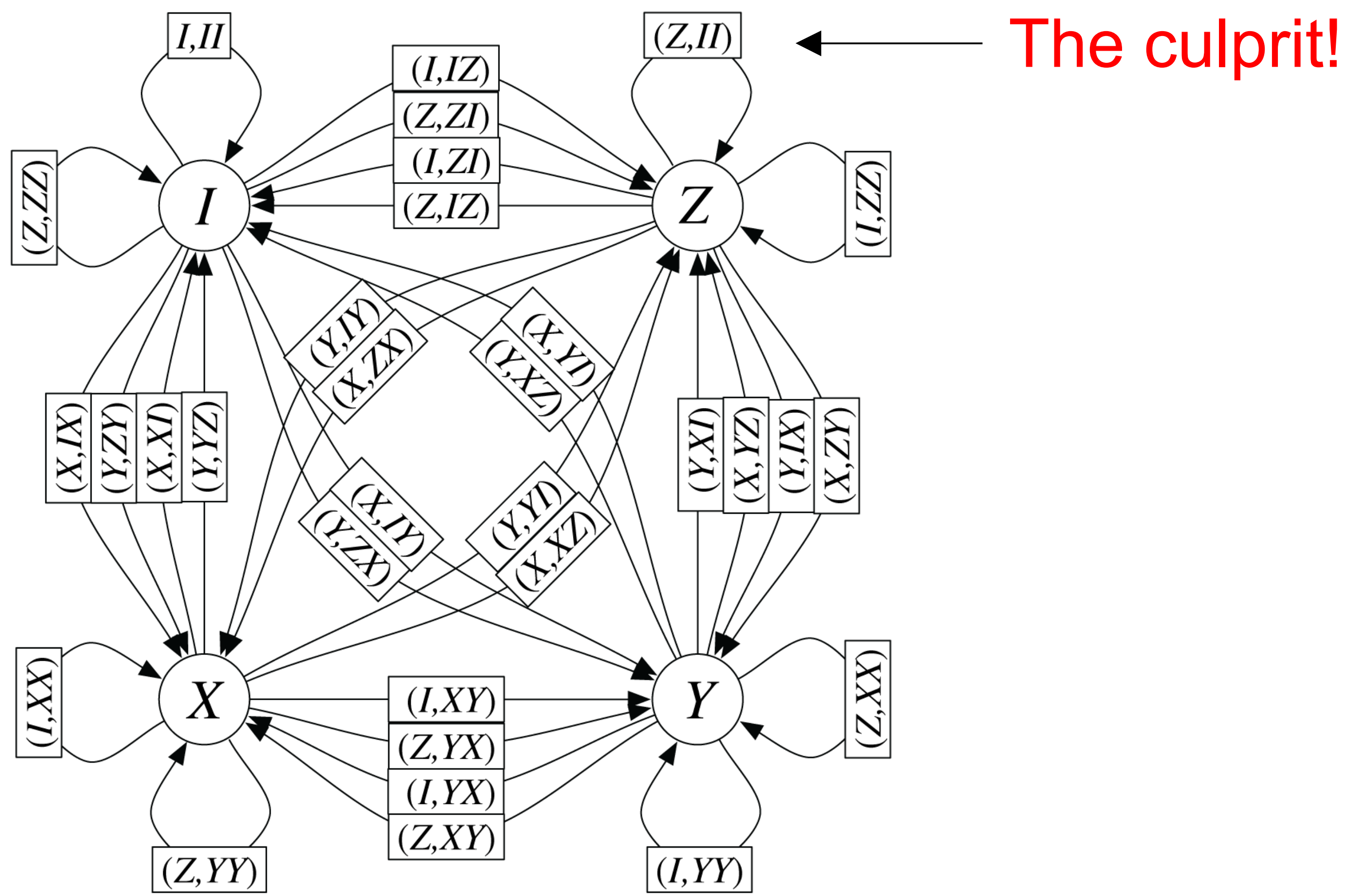


Catastrophic error propagation!

Catastrophicity (ctd.)

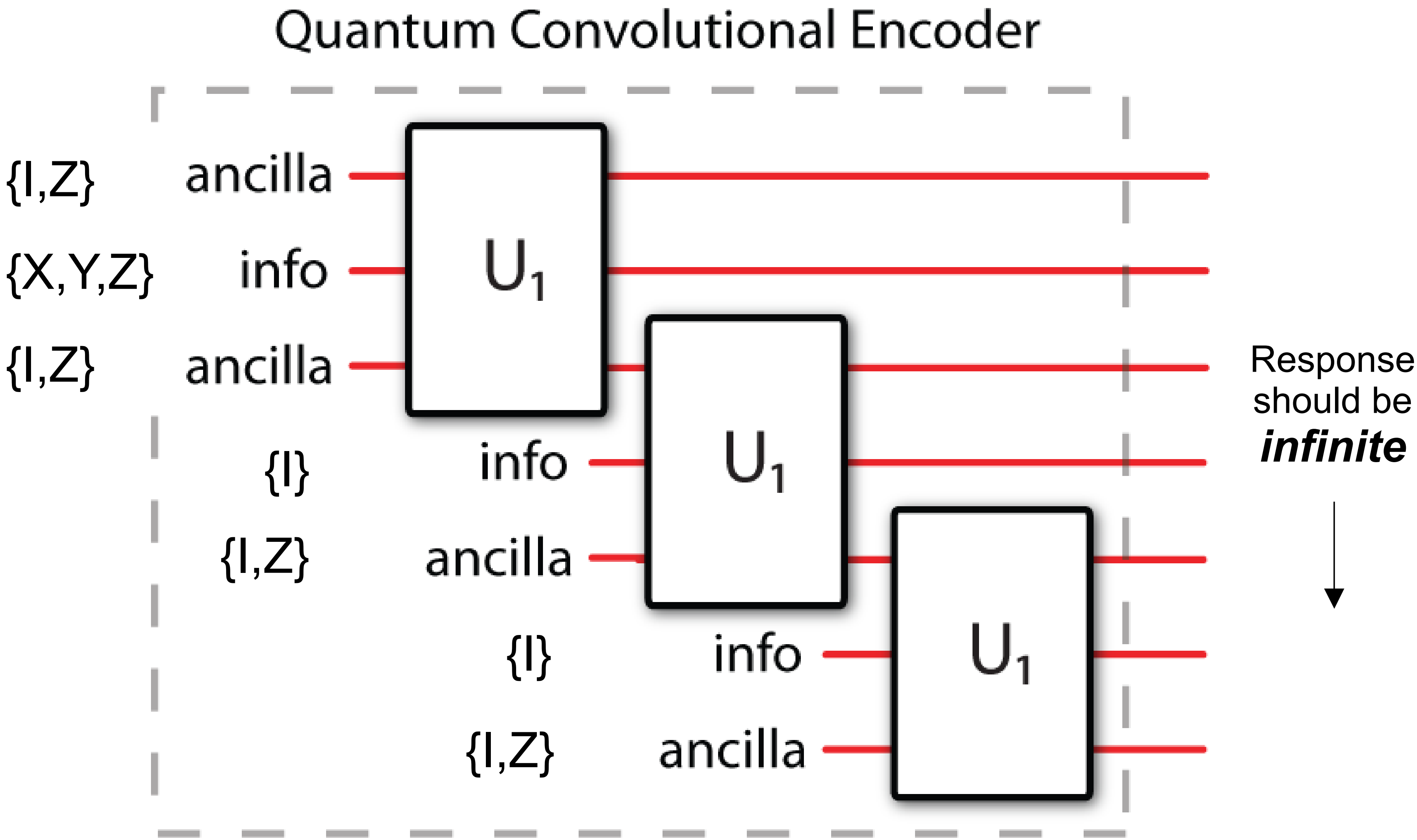
Check state diagram for cycles of zero physical weight with non-zero logical weight

(same as classical condition)



Viterbi. Convolutional codes and their performance in communication systems. *IEEE Trans. Comm. Tech.* (1971)

Recursiveness



A **recursive encoder** has an *infinite response* to a weight-one logical input

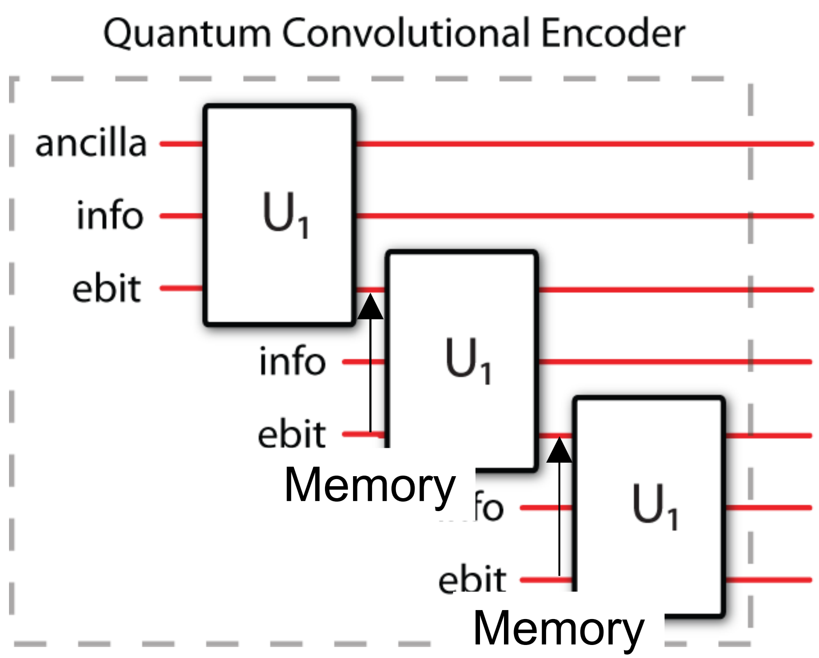
No-Go Theorem

Both **recursiveness** and **non-catastrophicity** are desirable properties for a quantum convolutional encoder when used in a quantum turbo code

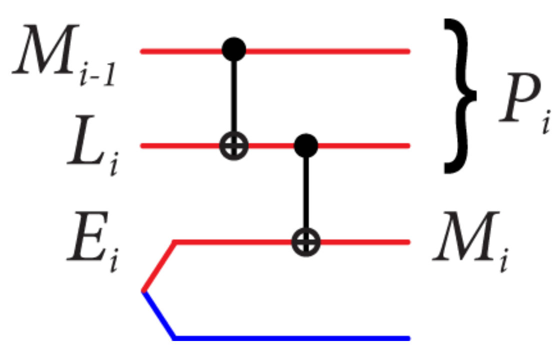
But a quantum convolutional encoder cannot have both!
(Theorem 1 of PTO)

D. Poulin, J.-P. Tillich, and H. Ollivier, "Quantum serial turbo-codes," *IEEE Transactions on Information Theory*, vol. 55, no. 6, pp. 2776– 2798, June 2009.

Idea: Add Entanglement



Example



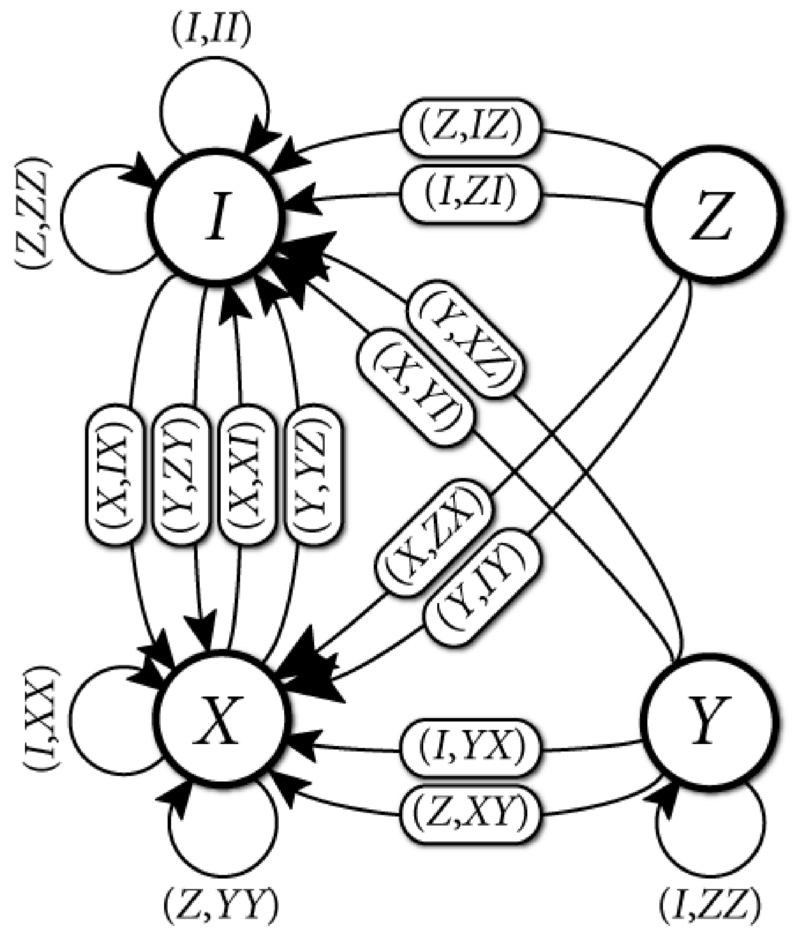
M. M. Wilde and M.-H. Hsieh, “Entanglement boosts quantum turbo codes,” In preparation.

State Diagram

Add an edge from one memory state to another
if a logical operator and identity on ebit connects them:

$$(M_{i-1} : L_i : I)U = (P_i : M_i)$$

State diagram
for EA example encoder

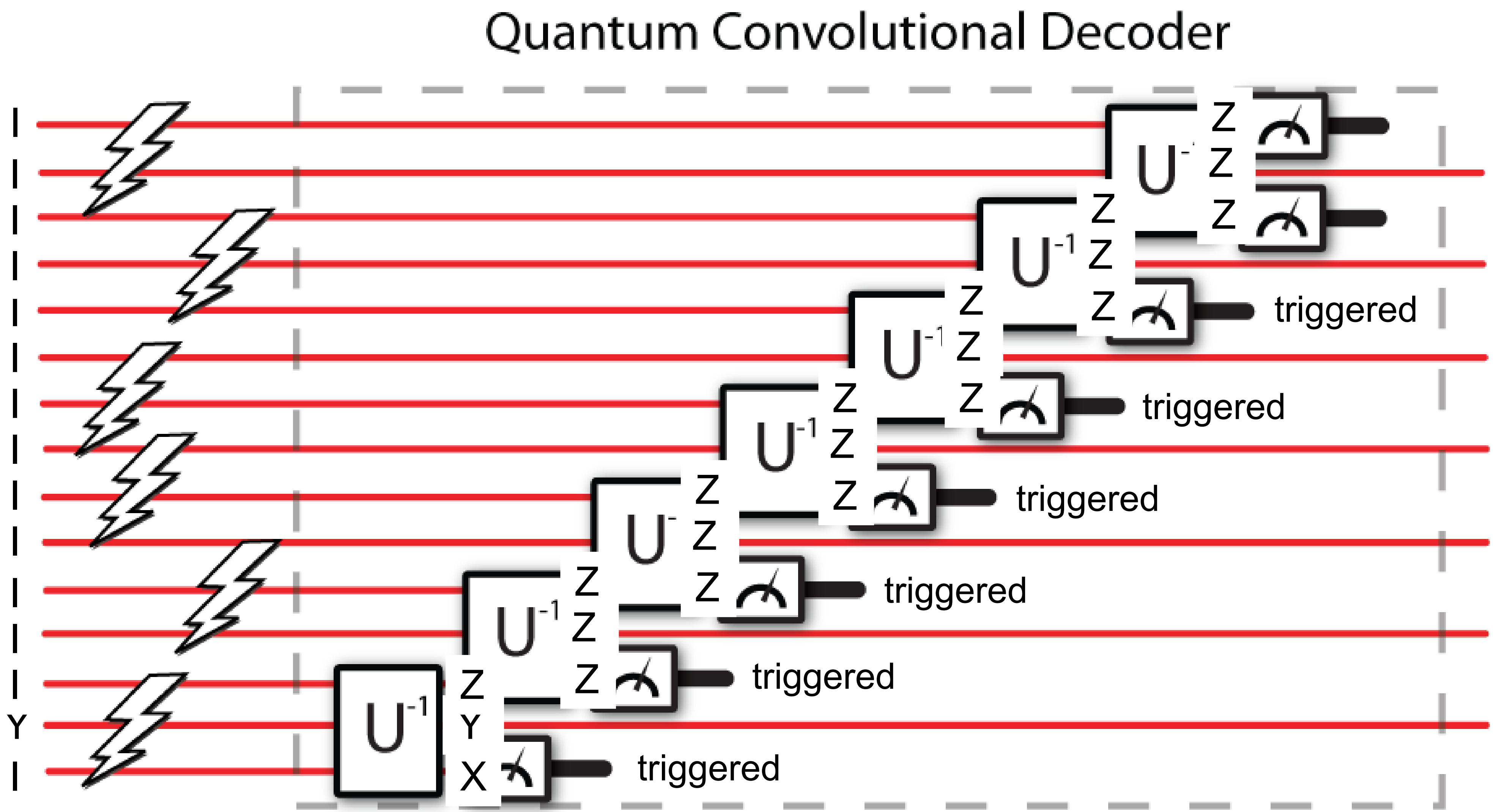


Tracks the flow of logical operators
through the convolutional encoder

Ebit removes half the edges!

M. M. Wilde and M.-H. Hsieh, "Entanglement boosts quantum turbo codes," In preparation.

Catastrophicity



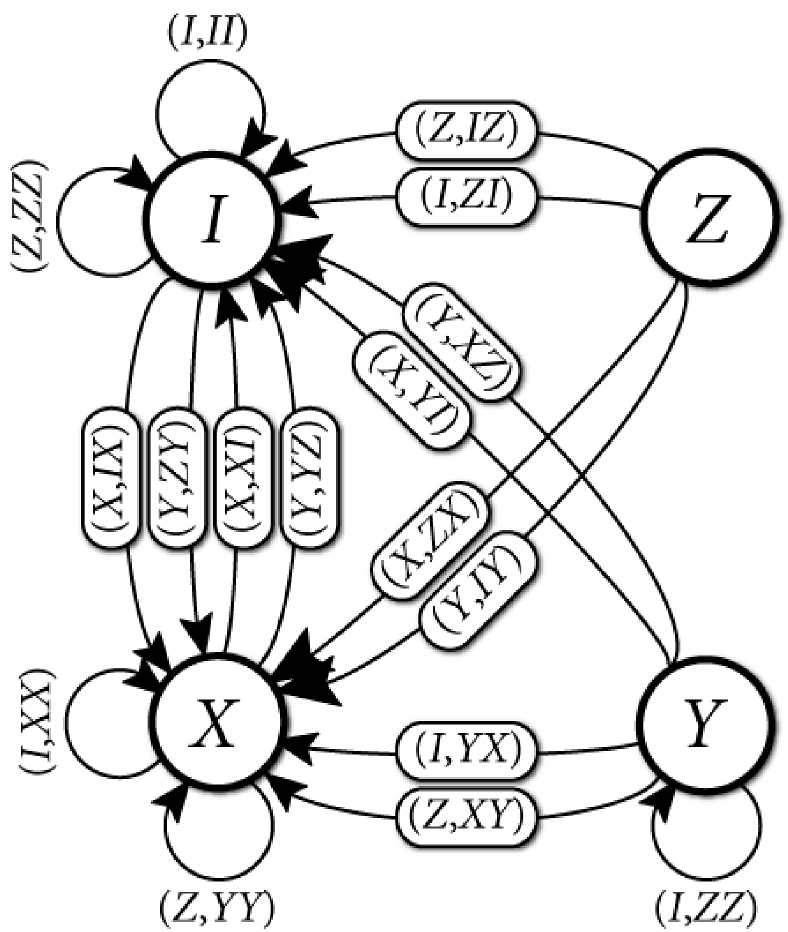
Catastrophic error propagation **eliminated**!

(Bell measurements detect Z errors)

M. M. Wilde and M.-H. Hsieh, "Entanglement boosts quantum turbo codes," In preparation.

Catastrophicity (ctd.)

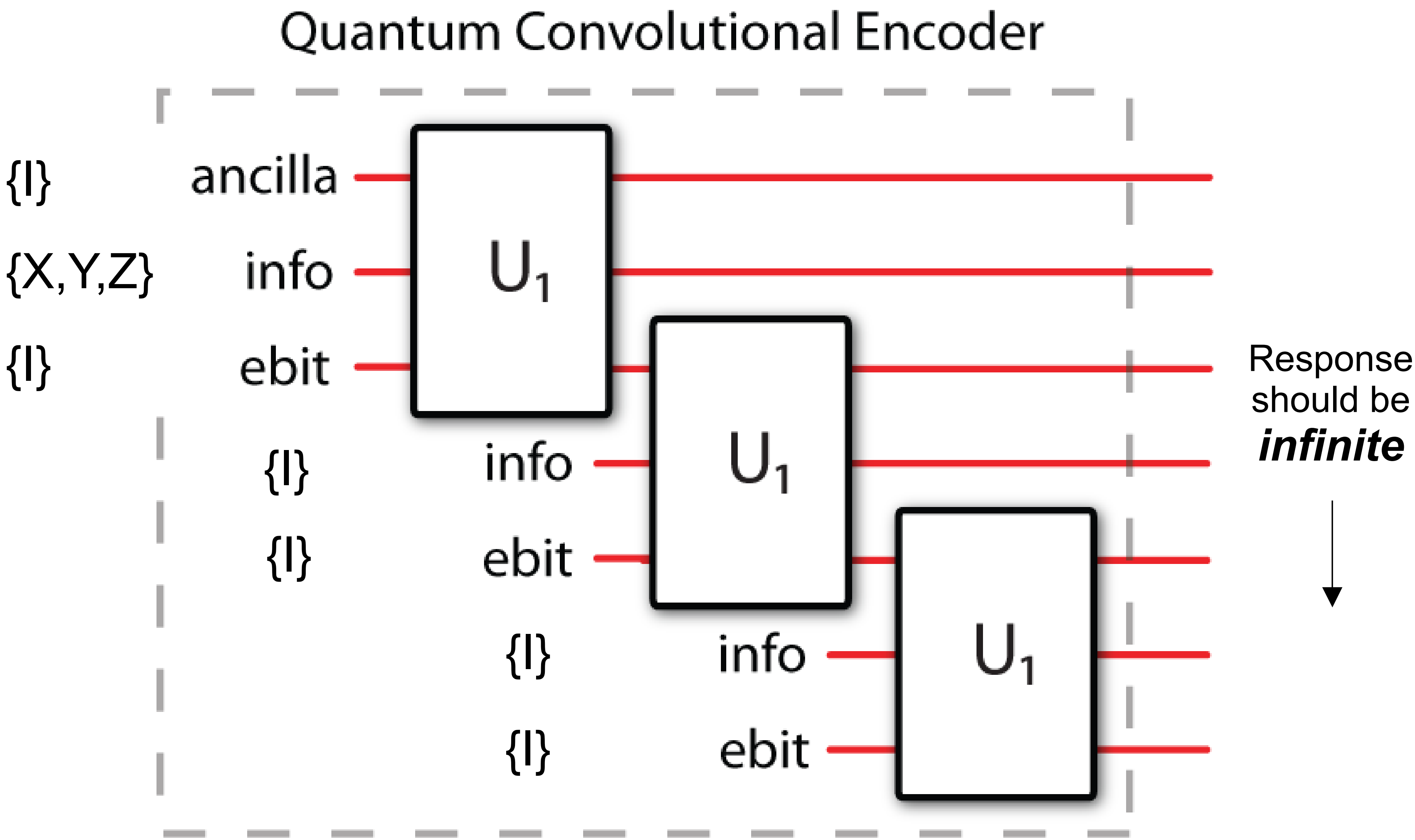
Check state diagram for cycles of zero physical weight with non-zero logical weight



Culprit gone!

M. M. Wilde and M.-H. Hsieh, “Entanglement boosts quantum turbo codes,” In preparation.

Recursiveness



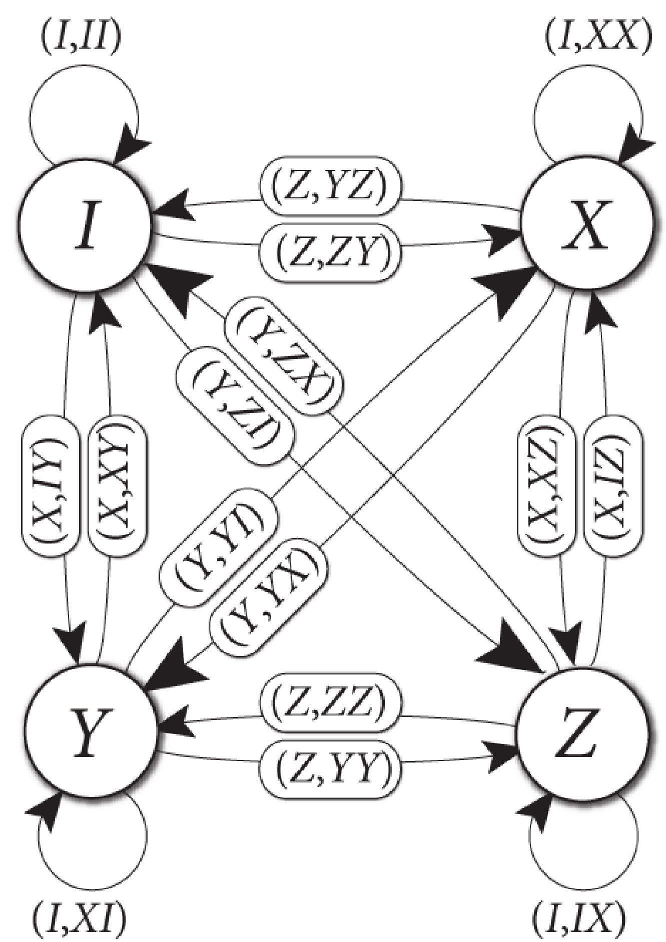
A **recursive encoder** has an *infinite response* to a weight-one logical input

M. M. Wilde and M.-H. Hsieh, "Entanglement boosts quantum turbo codes," In preparation.

Non-Catastrophic and Recursive Encoder

Memory in	Logical	Ebit		Memory out	Physical	Physical
Z	I	I		Z	I	X
I	Z	I		X	Z	Y
I	I	Z	$\rightarrow$	X	Y	Z
X	I	I		X	X	X
I	X	I		Y	I	Y
I	I	X		Y	X	Y

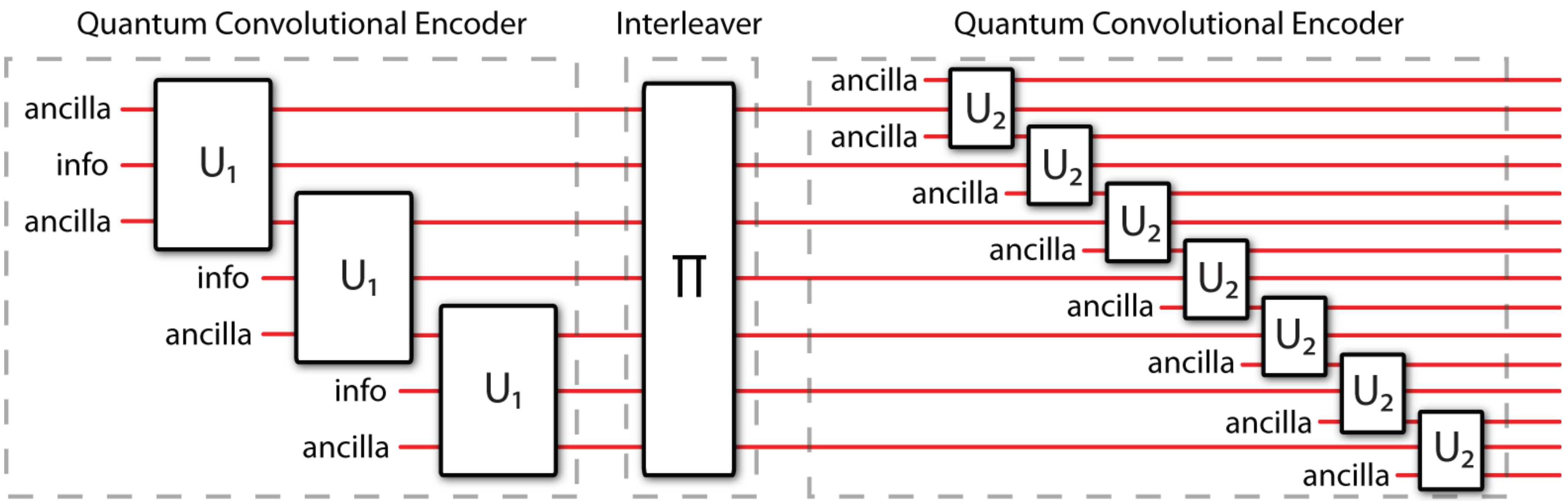
State Diagram



Entanglement-assisted encoders can satisfy both properties simultaneously!

M. M. Wilde and M.-H. Hsieh, “Entanglement boosts quantum turbo codes,” In preparation.

Quantum Turbo Codes

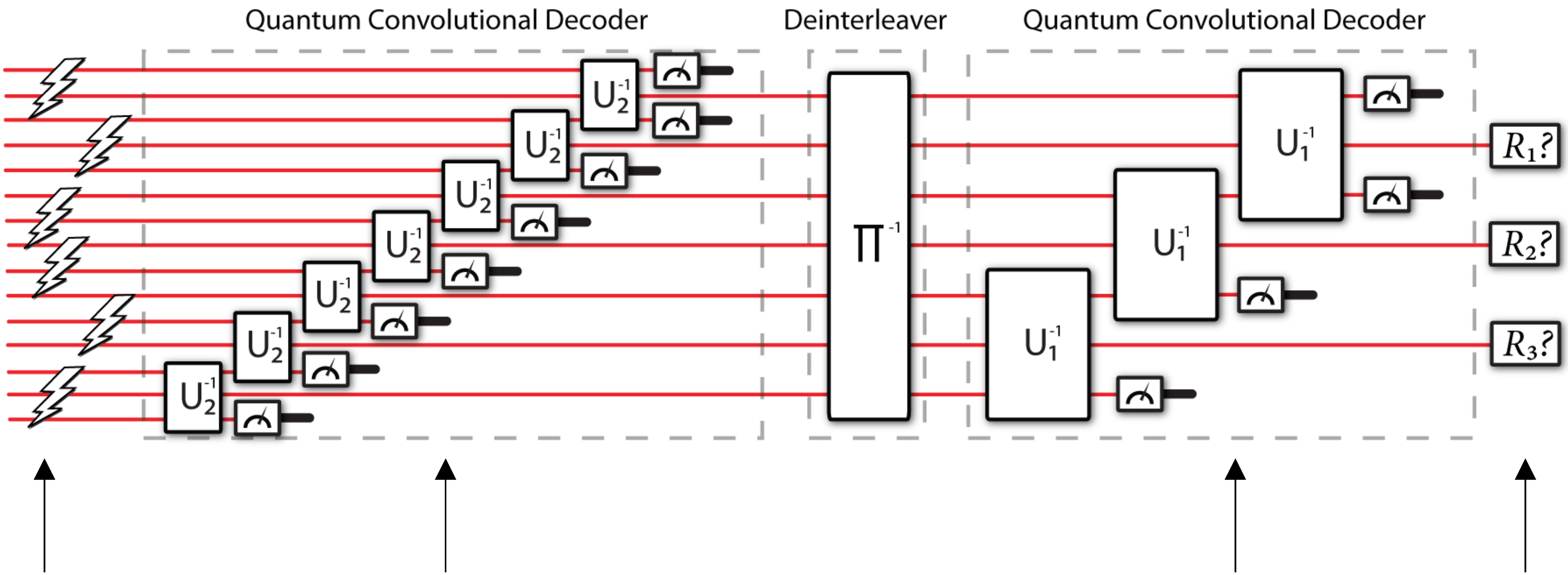


A **quantum turbo code** consists of two interleaved and serially concatenated quantum convolutional encoders

Performance **appears to be good** from the results of numerical simulations

D. Poulin, J.-P. Tillich, and H. Ollivier, "Quantum serial turbo-codes," *IEEE Transactions on Information Theory*, vol. 55, no. 6, pp. 2776– 2798, June 2009.

How to decode a Quantum Turbo Code?



Pauli error

Detect syndromes

Detect syndromes

Estimate errors

Do this last part with an iterative decoding algorithm

D. Poulin, J.-P. Tillich, and H. Ollivier, "Quantum serial turbo-codes," *IEEE Transactions on Information Theory*, vol. 55, no. 6, pp. 2776– 2798, June 2009.

Iterative Decoding

Three steps for each convolutional code:

1) backward recursion

2) forward recursion

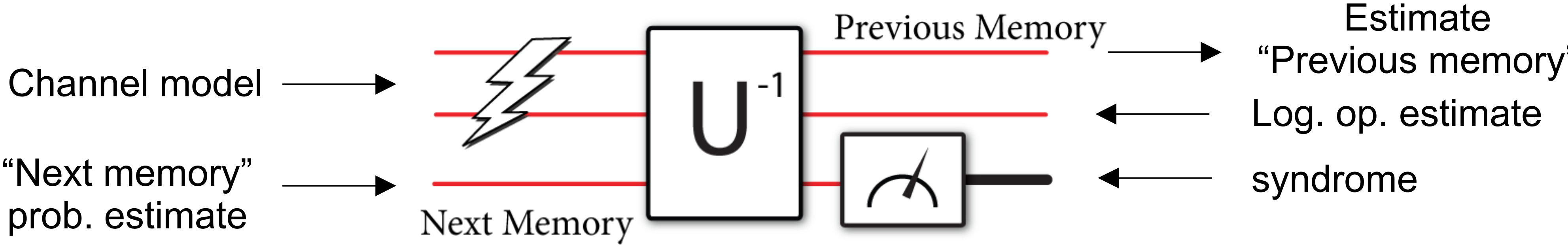
3) local update

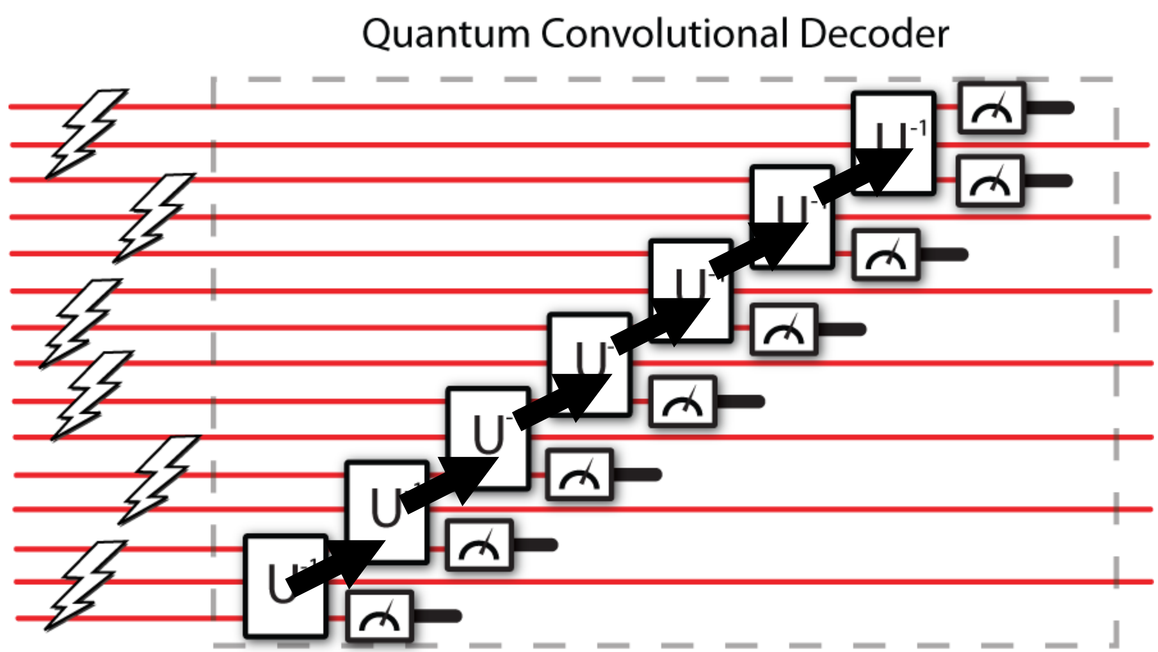
Decoders feed **probabilistic estimates** back and forth to each other until they converge on an estimate of the error

D. Poulin, J.-P. Tillich, and H. Ollivier, “Quantum serial turbo-codes,” *IEEE Transactions on Information Theory*, vol. 55, no. 6, pp. 2776– 2798, June 2009.

Iterative Decoding (Backward Recursion)

Use probabilistic estimates of “next” memory and logical operators, and the channel model and syndrome, to give soft estimate of “previous memory”:

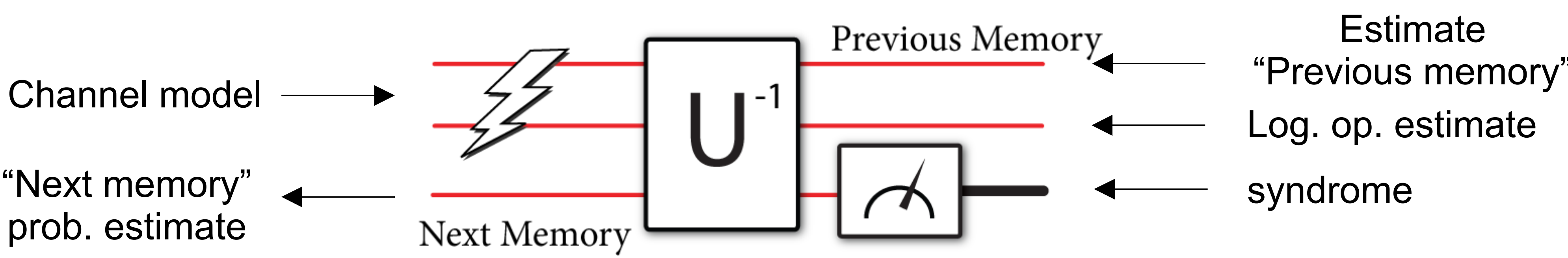


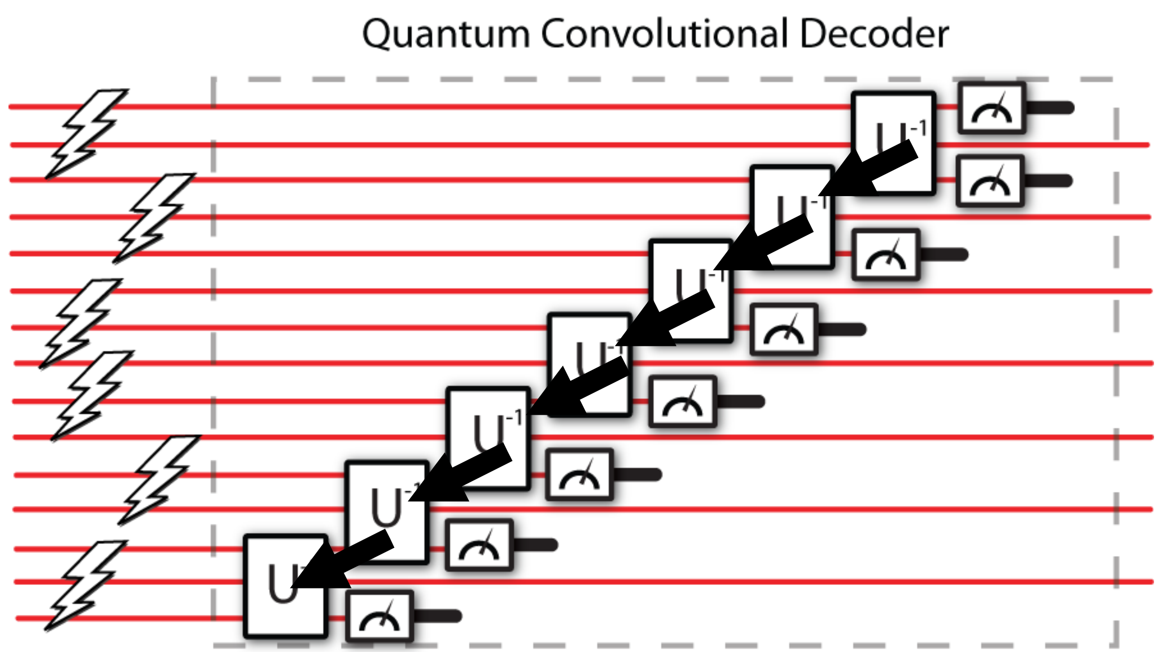


D. Poulin, J.-P. Tillich, and H. Ollivier, “Quantum serial turbo-codes,” *IEEE Transactions on Information Theory*, vol. 55, no. 6, pp. 2776– 2798, June 2009.

Iterative Decoding (Forward Recursion)

Use probabilistic estimates of "previous" memory and logical operators
and the channel model and syndrome,
to give soft estimate of "next memory":

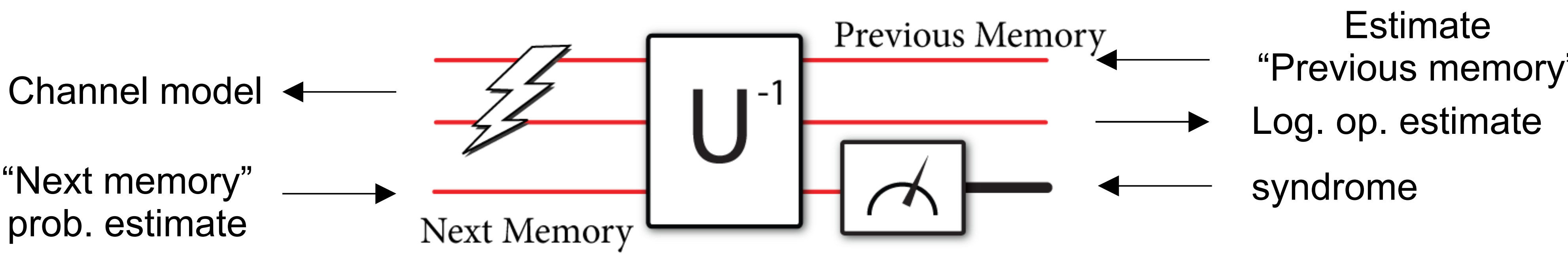




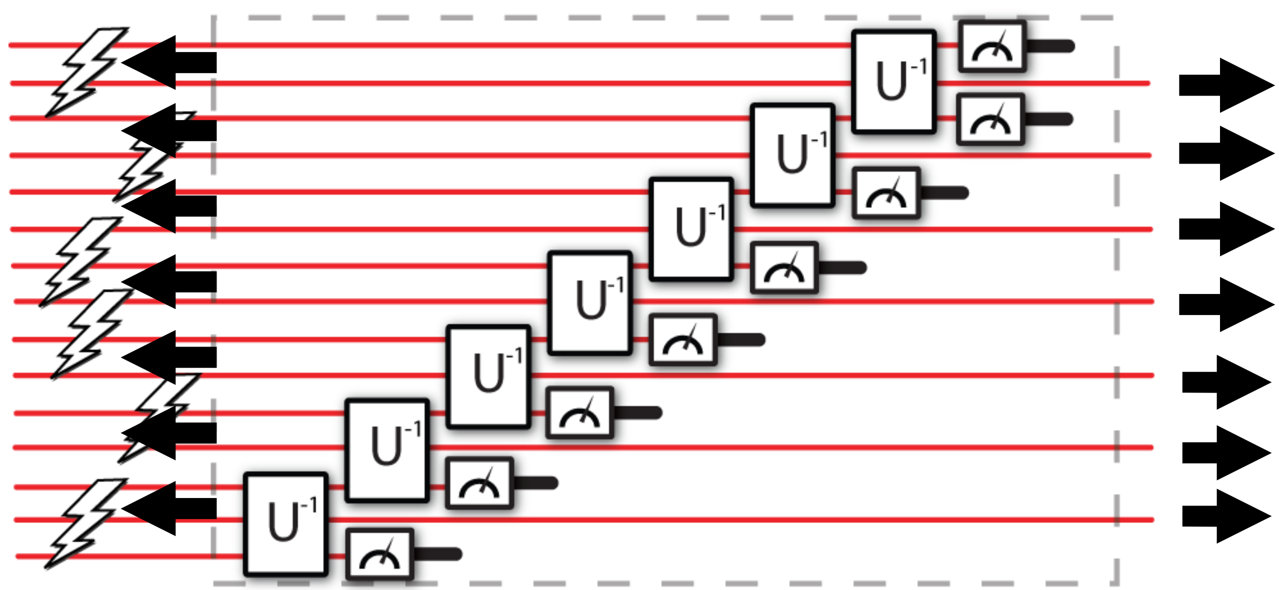
D. Poulin, J.-P. Tillich, and H. Ollivier, "Quantum serial turbo-codes,"
IEEE Transactions on Information Theory, vol. 55, no. 6, pp. 2776– 2798, June 2009.

Iterative Decoding (Local Update)

Use probabilistic estimates of
“previous” memory, “next memory”, and syndrome
to give soft estimate of logical ops and channel:

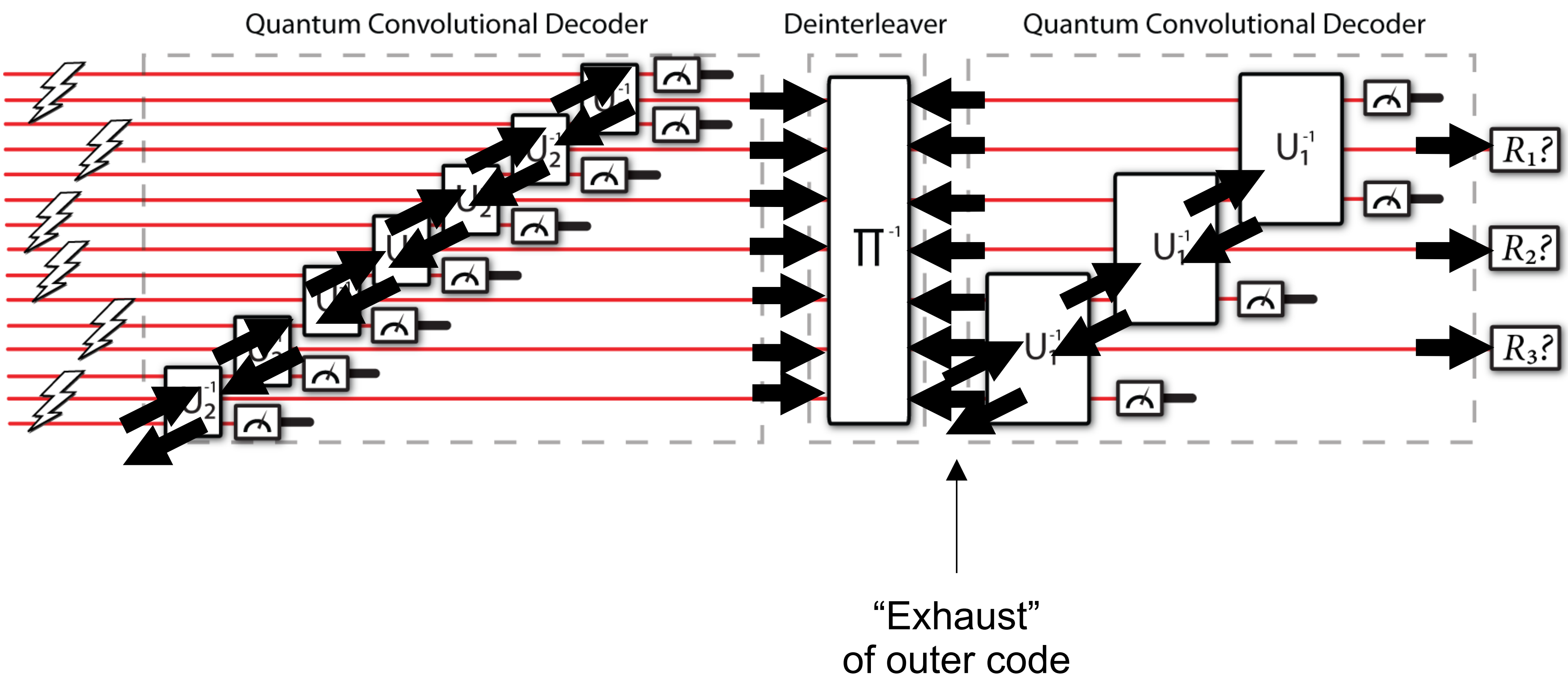


Quantum Convolutional Decoder



D. Poulin, J.-P. Tillich, and H. Ollivier, “Quantum serial turbo-codes,”
IEEE Transactions on Information Theory, vol. 55, no. 6, pp. 2776– 2798, June 2009.

Iterative Decoding of a Quantum Turbo Code



Iterate this procedure
until convergence or
some maximum number of iterations

D. Poulin, J.-P. Tillich, and H. Ollivier, "Quantum serial turbo-codes," *IEEE Transactions on Information Theory*, vol. 55, no. 6, pp. 2776– 2798, June 2009.

Simulations

Selected an encoder **randomly**
with one information qubit, two ancillas, and three memory qubits

Non-catastrophic and **quasi-recursive**

Distance spectrum:

$$11x^5 + 47x^6 + 253x^7 + 1187x^8 + 6024x^9 + 30529x^{10} + 153051x^{11} + 771650x^{12}$$

Serial concatenation with itself gives
a rate 1/9 quantum turbo code

Replacing both ancillas with ebits gives EA encoder

Non-catastrophic and **recursive**

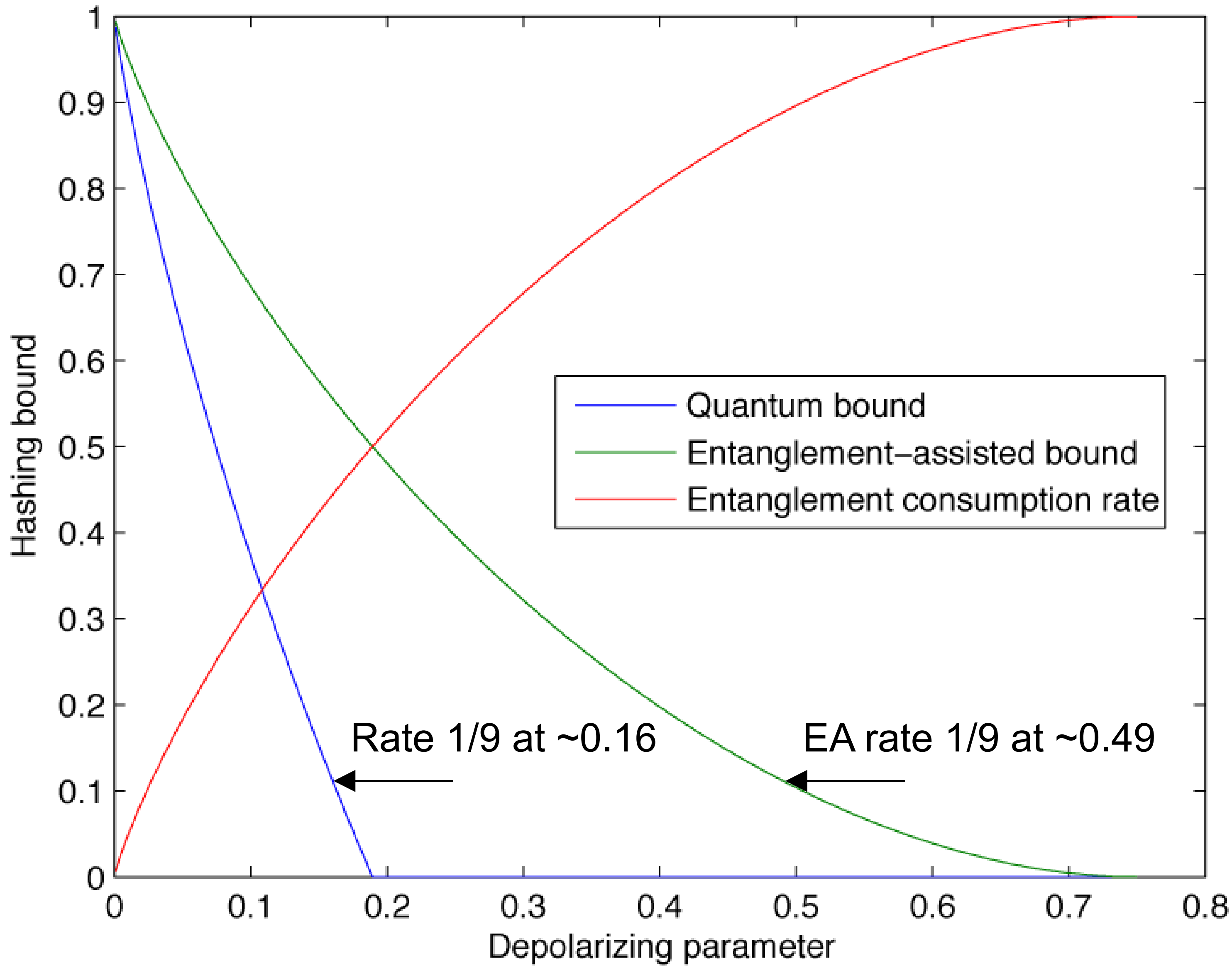
Distance spectrum improves dramatically:

$$2x^9 + x^{10} + 5x^{11} + 8x^{12}$$

Serial concatenation with itself gives
a rate 1/9 quantum turbo code
with 8/9 entanglement consumption rate

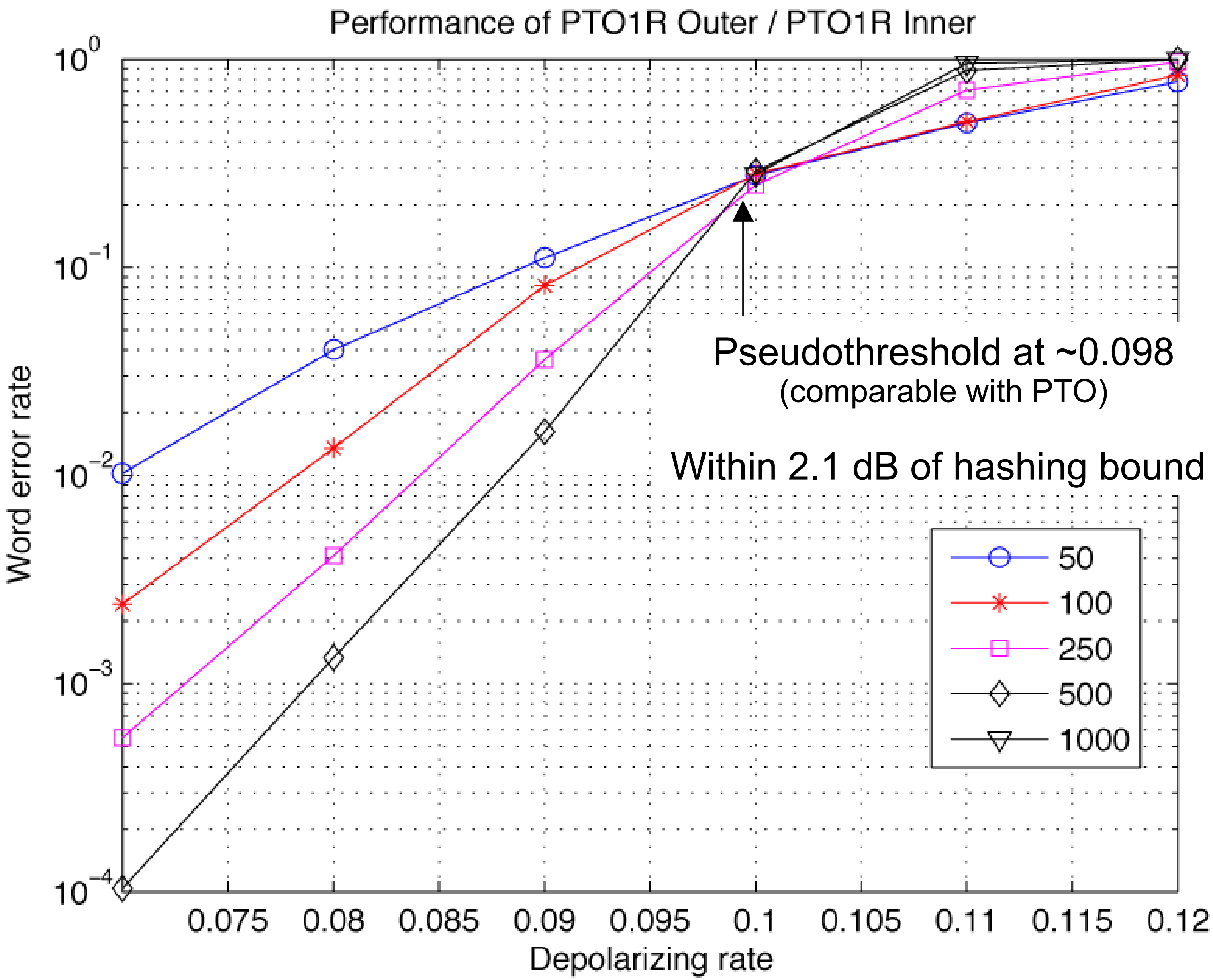
M. M. Wilde and M.-H. Hsieh, "Entanglement boosts quantum turbo codes," In preparation.

Compare with the Hashing Bounds



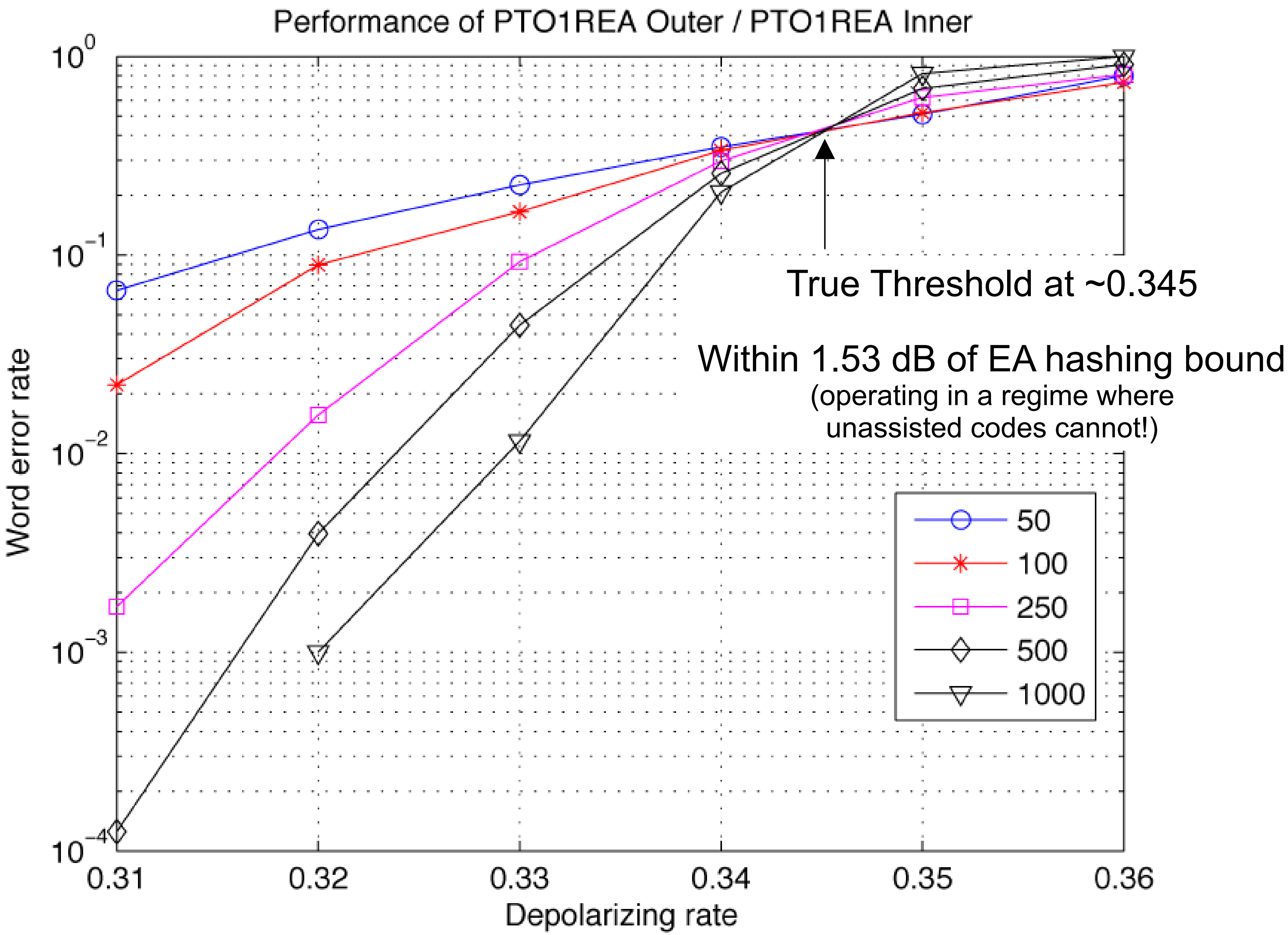
Bennett *et al*., "Entanglement-assisted classical capacity," (2002)
Devetak *et al*., "Resource Framework for Quantum Shannon Theory (2005)

Unassisted Turbo Code



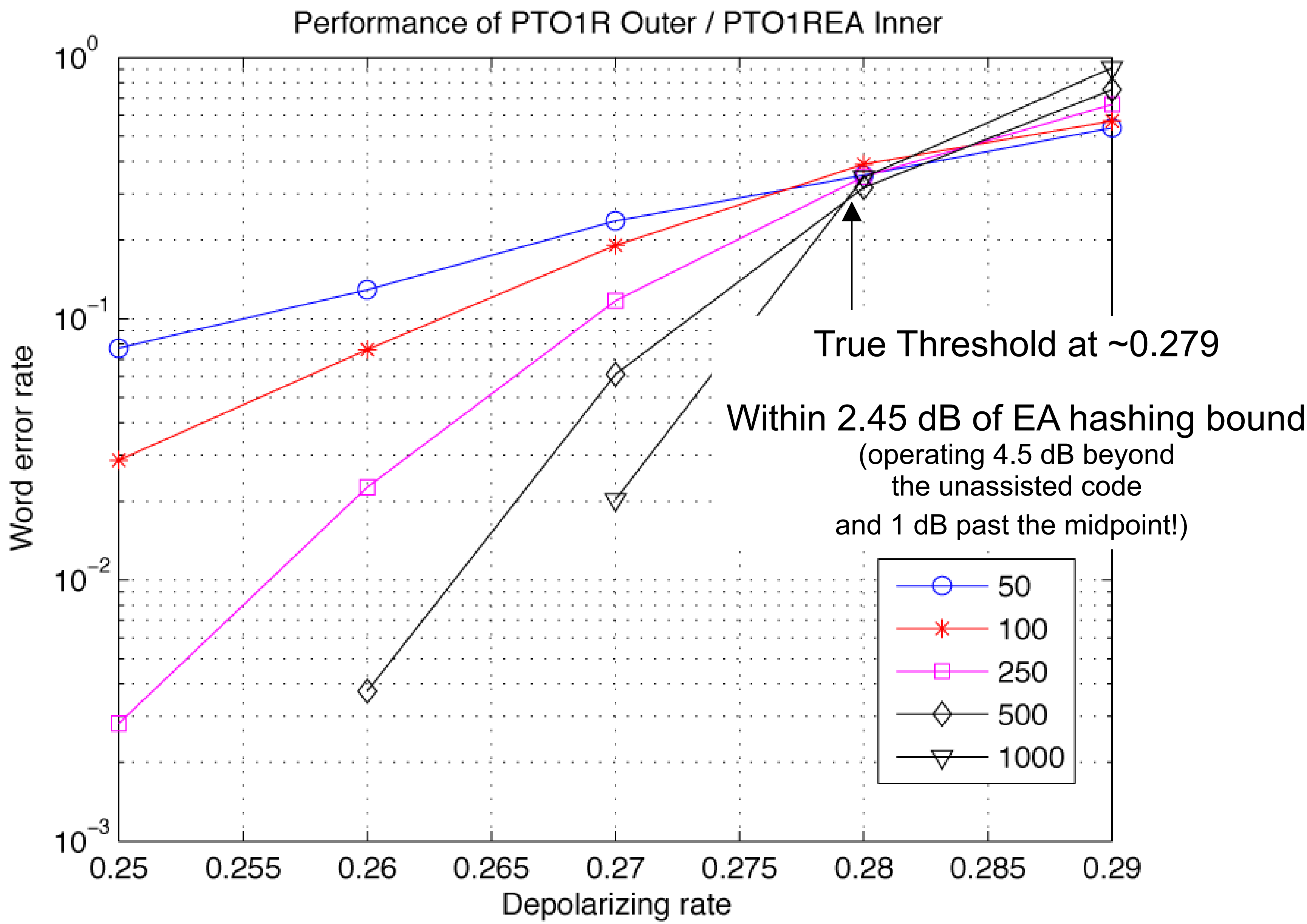
M. M. Wilde and M.-H. Hsieh, “Entanglement boosts quantum turbo codes,” In preparation.

Fully Assisted Turbo Code



M. M. Wilde and M.-H. Hsieh, "Entanglement boosts quantum turbo codes," In preparation.

"Inner" Entanglement Assisted Turbo Code



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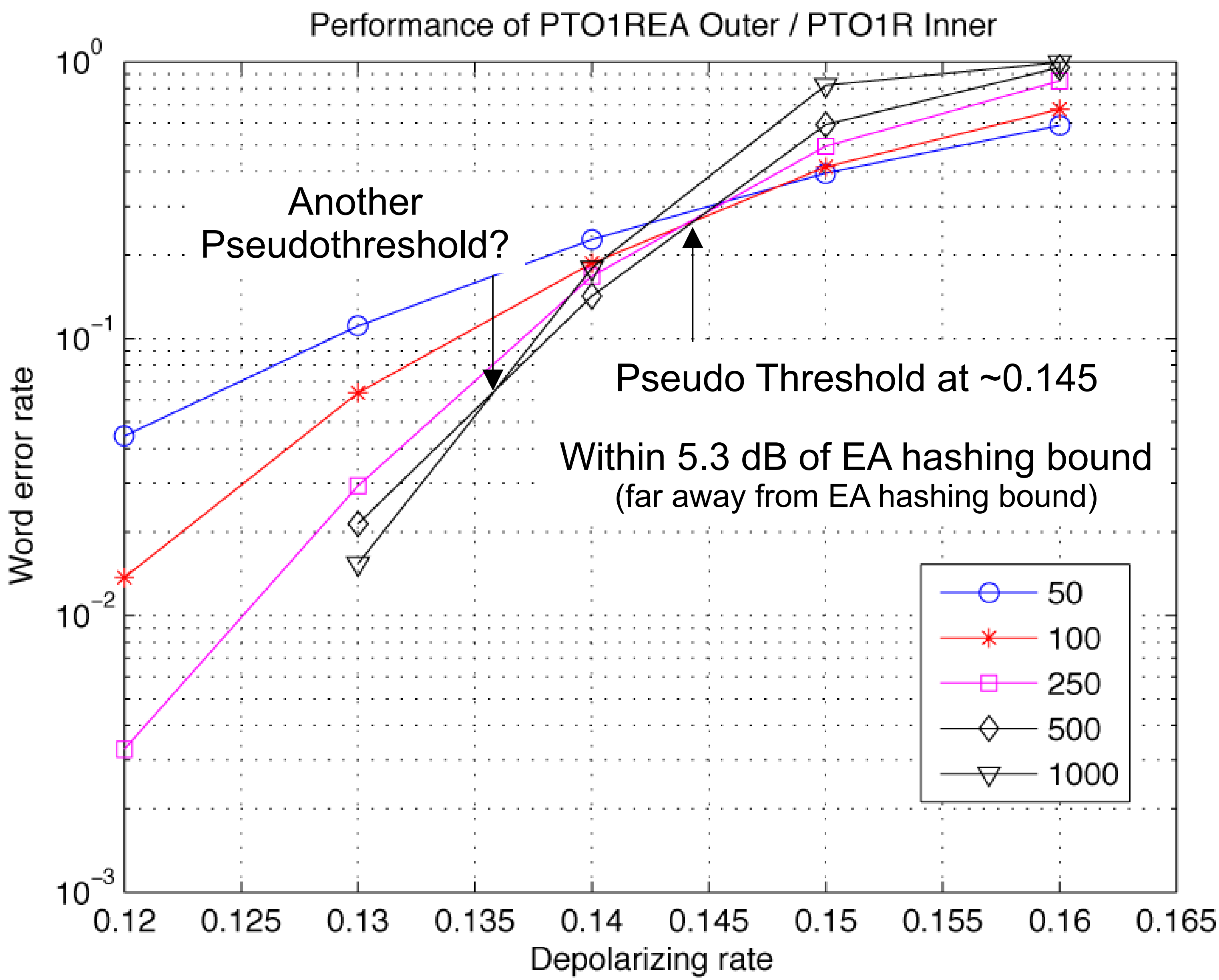
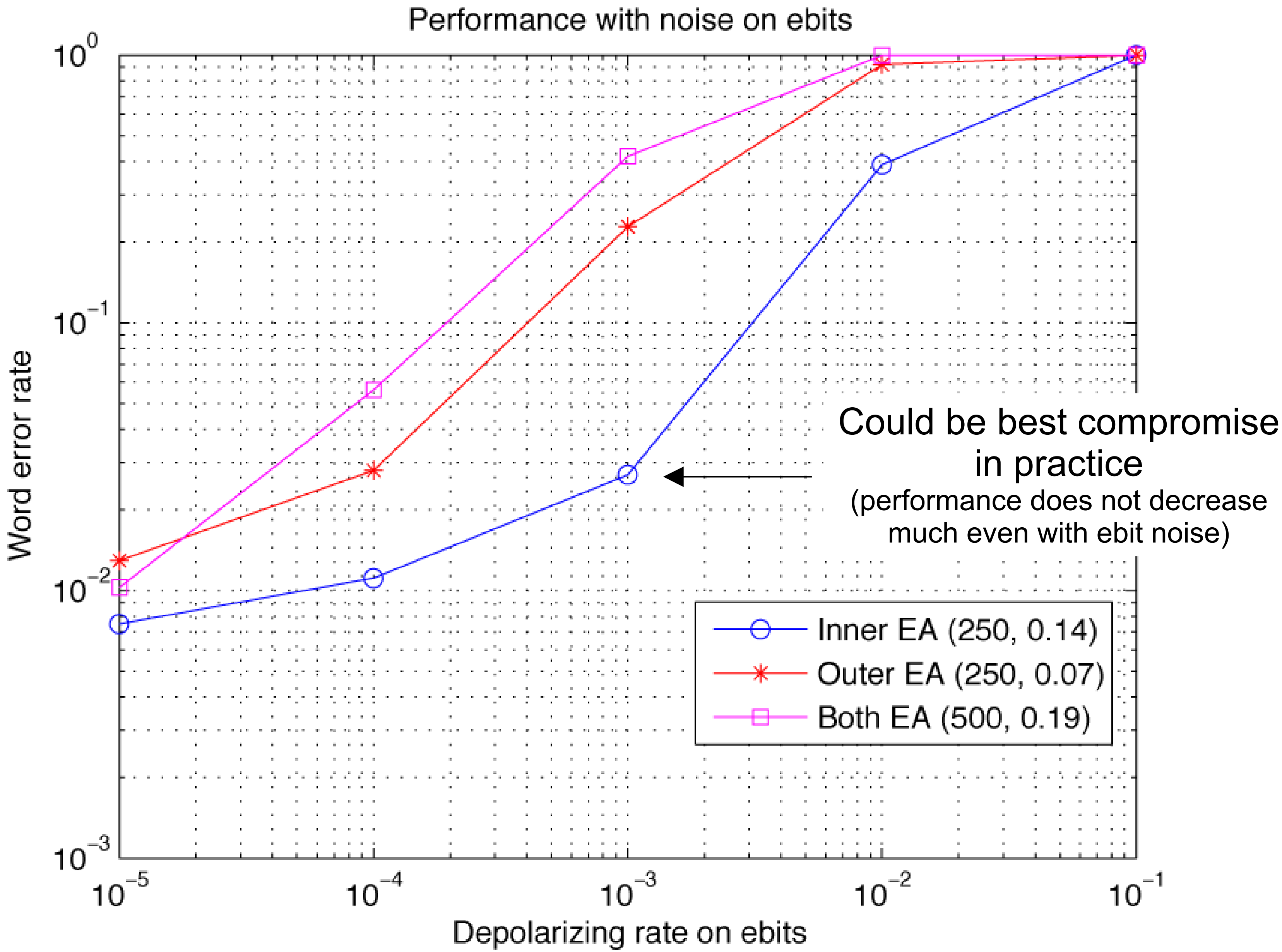
"Outer" Entanglement Assisted Turbo Code



Quasi-recursiveness does not explain good performance of unassisted code!

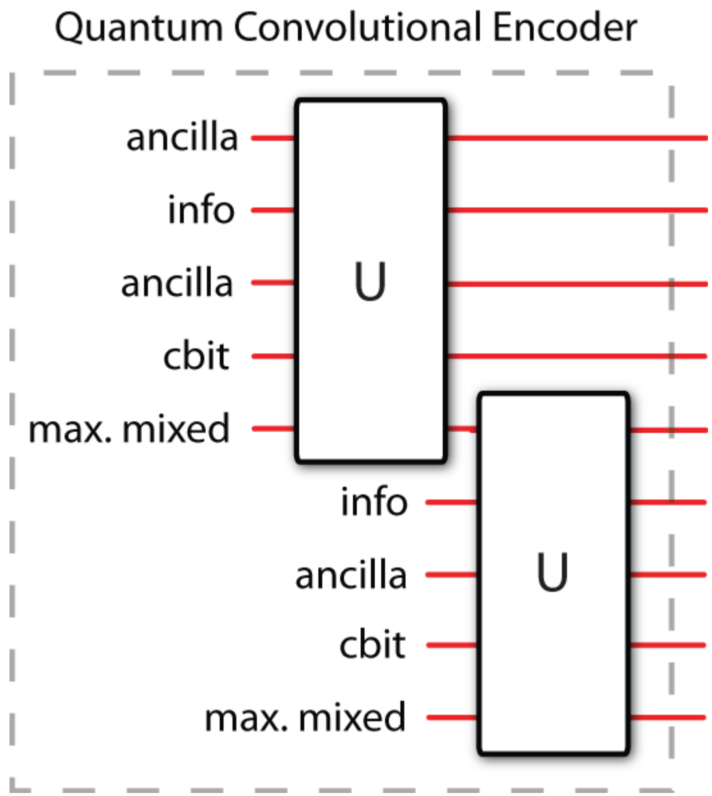
M. M. Wilde and M.-H. Hsieh, "Entanglement boosts quantum turbo codes," In preparation.

Adding Noise to Bob's Share of the Ebits



M. M. Wilde and M.-H. Hsieh, "Entanglement boosts quantum turbo codes," In preparation.

No-Go Theorem for Subsystem or Classically-Enhanced Codes



Encoder of the above form cannot be **recursive** and **non-catastrophic**

Proof: Consider recursive encoder.
Change gauge qubits and cbits to ancillas (preserves recursiveness)
Must be catastrophic (by PTO)
Change ancillas back to gauge qubits and cbits (preserves catastrophicity).

M. M. Wilde and M.-H. Hsieh, “Entanglement boosts quantum turbo codes,” In preparation.

Conclusion

- Entanglement gives both a theoretical and practical boost to quantum turbo codes

- Recursiveness is essential to good performance of the assisted code (not mere quasi-recursiveness)

- No-Go Theorem for subsystem and classically-enhanced encoders

Open question: Find an EA turbo code with positive catalytic rate that outperforms a PTO encoder

Open question: Can turbo encoders with logical qubits, cbits, and ebits come close to achieving trade-off capacity rates?