Trade-off capacities of the quantum Hadamard channels

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Joint work with
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Quantum Shannon Theory

Channel Capacities

- Goal: Transmit information over noisy quantum channels

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- General case: Optimization over arbitrarily many parallel uses

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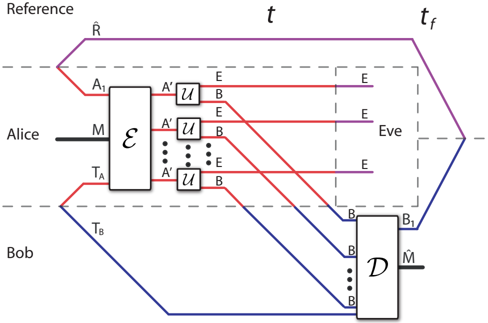
- Quantum channels have different types of capacities
- General case: Optimization over arbitrarily many parallel uses
- Single-Letter case: Optimization over only a single channel use

Quantum Shannon Theory

Quantum Channels



- $\mathcal{U}$: Quantum channel
  Input Alice: A', Output Bob: B, Environment Eve: E

Quantum Shannon Theory

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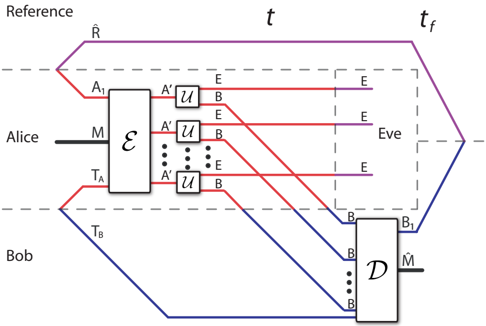

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  Quantum state: A_1, B_1, Purifying system: R
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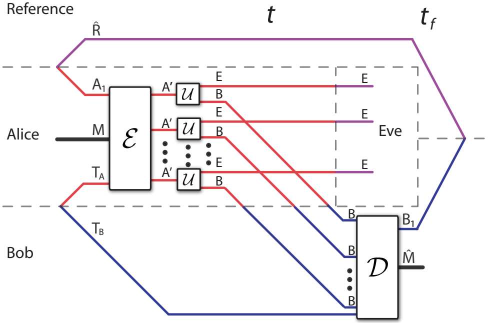

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- Encoder: $\mathcal{E}$, Decoder: $\mathcal{D}$

Capacity Regions

- Important special cases:

[1] I. Devetak and P. W. Shor. Communications in Mathematical Physics, 256:287303, 2005.

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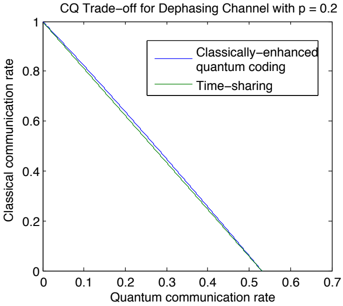

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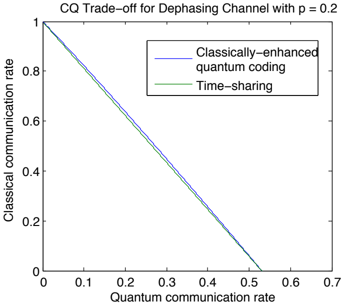

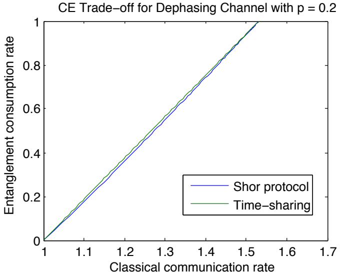

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- Simultaneous communication of classical and quantum information with limited entanglement-assistance[3]

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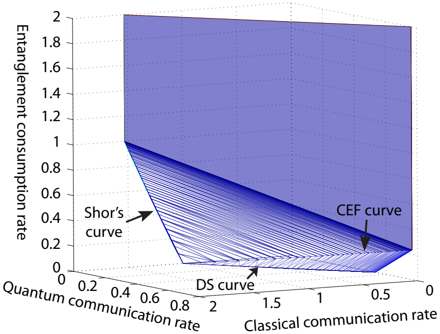
- Simultaneous communication of classical and quantum information with limited entanglement-assistance[3]
- Optimal rates require optimization over arbitrarily many parallel channel uses
- Single-letterization of two special cases implies whole triple trade-off region[3]



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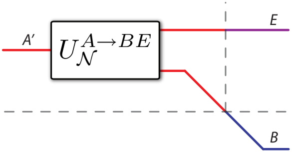
Quantum Hadamard Channels

Definition

- Quantum channel $\mathcal{N}^{A\to B}$ has isometric extension $U_{\mathcal{N}}^{A\to BE}$



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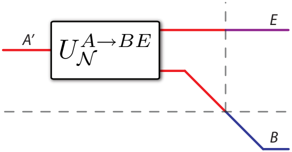

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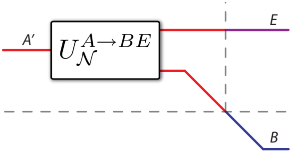

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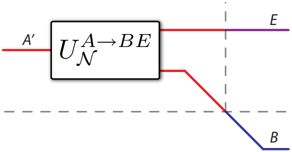

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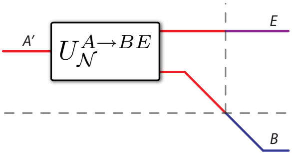

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- Hadamard channel:
  Complementary channel is **entanglement-breaking**

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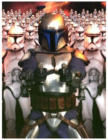
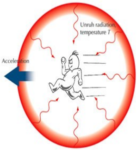


- Unruh Channel $\mathcal{N}_U$: Arises in QFT
  - Mathematical Structure: Block Diagonal[a,b]
  - $\mathcal{N}_U = \bigoplus_{N=1}^{\infty} p_N \mathcal{N}_N$



[a] K. Brádler, P. Hayden, and P. Panangaden. JHEP, 2009(08):074, 2009.

[b] K. Brádler. arXiv:0903.1638, 2009.

Capacity Region for Hadamard Channels

Degradable Channels

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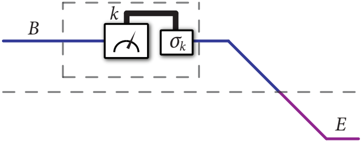
- Hadamard channels are degradable
- Structure of degrading map:

$$\mathcal{T} = \mathcal{T}_2 \circ \mathcal{T}_1$$

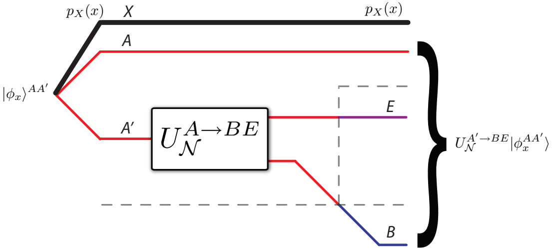
- $\mathcal{T}_1$: von Neumann measurement
- $\mathcal{T}_2$: conditional state preparation
- Gives a single-letter capacity formula



Capacity Region for Hadamard Channels

Single-letterization

Input state ρ:



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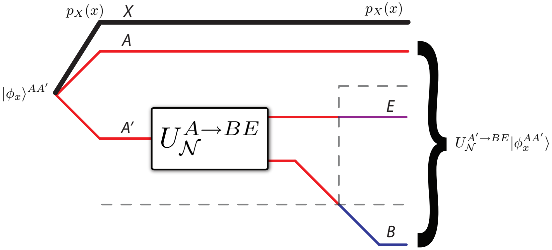

- Classical-Quantum trade-off curve:
  $f_\lambda(\mathcal{N}) = \max_\rho I(X;B)_{\mathcal{N}(\rho)} + \lambda I(A\rangle BX)_{\mathcal{N}(\rho)}$

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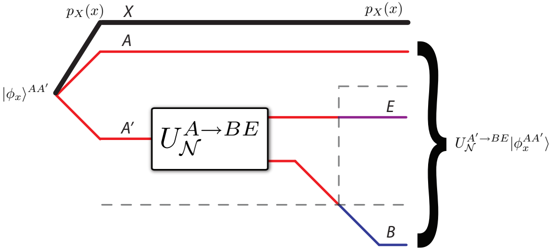

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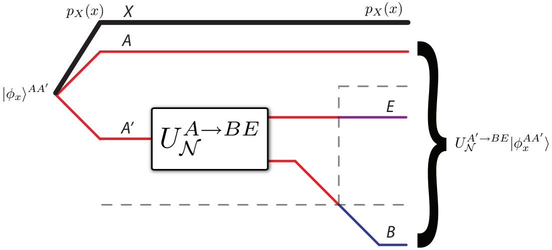

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- Single-letterization: $f_\lambda(\mathcal{N}^{\otimes k}) = kf_\lambda(\mathcal{N})$, $g_\lambda(\mathcal{N}^{\otimes k}) = kg_\lambda(\mathcal{N})$

Trade-off curves for the Dephasing Channel

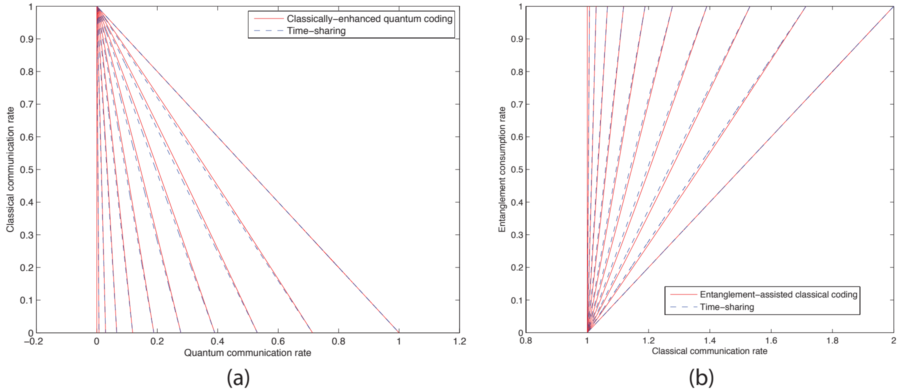

Figure: Parametrization for qubit p-dephasing channel, with $p = 0, 0.1, \ldots, 0.9, 1$: $\mathcal{N} = (1-p)I + p\Delta$, I identity channel, Δ Completely dephasing channel

Trade-off curves for $1 \to N$ Cloning Channels

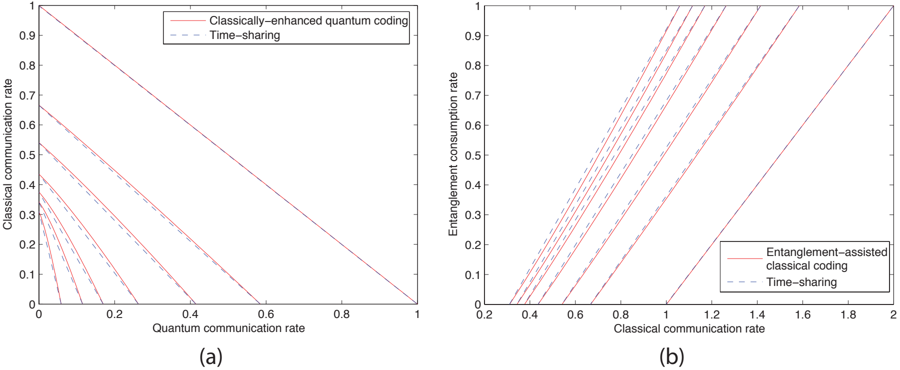

Figure: Parametrization for $1 \to N$ Cloning Channels, with $N = 1, 2, 3, 5, 8, 12, 24$

Parametrization of the Triple Trade-Off Region

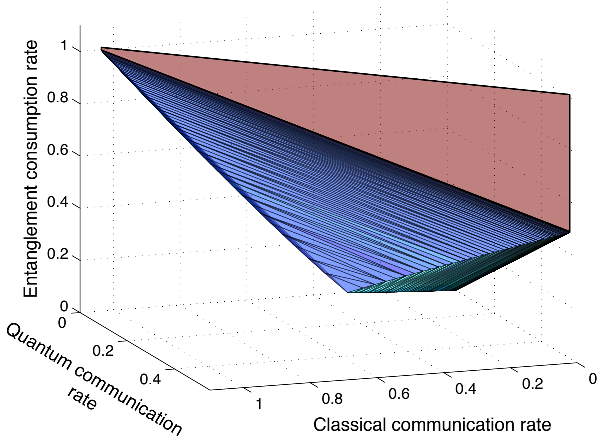

Figure: Unruh channel with acceleration parameter $z = 0.95$: $z = 0$ identity channel, $z \to 1$ infinite acceleration

Measuring Improvement over Time-Sharing

- Want a measure of relative improvement
- Ratio of Area under curves

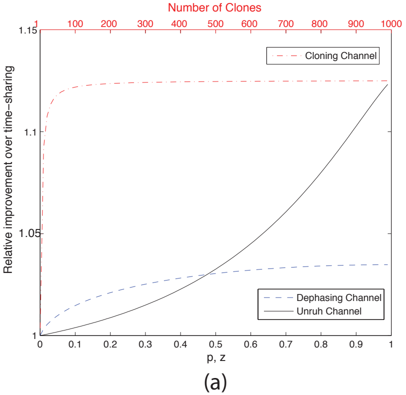

Figure: Relative improvement for (a) classical-quantum and (b) entanglement-assisted classical trade-off

Summary

- Single-letterization of the whole triple trade-off region for Hadamard channels
- Optimal coding strategy for Hadamard channels completely understood
- Parametrization of whole 3D region for 3 natural subclasses of Hadamard channels
- Introduction of measure of relative gain
- Important to consider optimal coding strategy to use resources to full potential