Trade-off capacities of the quantum Hadamard channels

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Joint work with
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Goal: Transmit information over noisy quantum channels
Quantum Shannon Theory

Channel Capacities

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Quantum Shannon Theory

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- General case: Optimization over arbitrarily many parallel uses
Quantum Shannon Theory

Channel Capacities

- **Goal**: Transmit information over noisy quantum channels
- **Capacity**: is the maximum rate of reliable communication
- **Measured**: in bits per channel use
- **Quantum channels**: have different types of capacities
- **General case**: Optimization over arbitrarily many parallel uses
- **Single-Letter case**: Optimization over only a single channel use
Quantum Shannon Theory

Quantum Channels

- $\mathcal{U}$: Quantum channel
  - Input Alice: $A'$, Output Bob: $B$, Environment Eve: $E$
Quantum Shannon Theory
Quantum Channels

- \( \mathcal{U} \): Quantum channel
  - Input Alice: \( A' \), Output Bob: \( B \), Environment Eve: \( E \)
- Classical message: \( M, \hat{M} \)
- Quantum state: \( A_1, B_1 \), Purifying system: \( R \)
- Shared Entanglement: \( T_A, T_B \)
Quantum Shannon Theory
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- Encoder: $\mathcal{E}$, Decoder: $\mathcal{D}$
Capacity Regions

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  - Entanglement-assisted classical communication\(^2\)

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\(^1\) I. Devetak and P. W. Shor. Communications in Mathematical Physics, 256:287303, 2005.

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Triple Trade-Off

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- Simultaneous communication of classical and quantum information with limited entanglement-assistance\(^3\)
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Triple Trade-Off

- Simultaneous communication of classical and quantum information with limited entanglement-assistance
- Optimal rates require optimization over arbitrarily many parallel channel uses
- Single-letterization of two special cases implies whole triple trade-off region

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Quantum Hadamard Channels

Definition

- Quantum channel $\mathcal{N}^{A\rightarrow B}$ has isometric extension $U_{\mathcal{N}}^{A\rightarrow BE}$

- $\mathcal{N}^c$: Complementary Channel
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- Entanglement-Breaking channel: Outputs separable state if input entangled state
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- Hadamard channel: Complementary channel is entanglement-breaking
Quantum Hadamard Channels

Examples

- Generalized dephasing channel represents loss of coherence
  - Pure basis state unaffected, superpositions get mixed
Quantum Hadamard Channels

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- Universal $1 \rightarrow N$ cloning channel $\mathcal{N}_N$
  - Input: 1 qubit, Output: $N$ approximate copies
  - Maximal Copy Fidelity, Independent of Input
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  - Input: 1 qubit, Output: N approximate copies
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- Unruh Channel $\mathcal{N}_U$: Arises in QFT
  - Mathematical Structure: Block Diagonal$^{ab}$
    - $\mathcal{N}_U = \bigoplus_{N=1}^{\infty} p_N \mathcal{N}_N$

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Capacity Region for Hadamard Channels

Degradable Channels

- Degradable channel:
  
  *Bob can simulate Eve*
Degradable Channels

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- Degrading map $\mathcal{T}: \mathcal{N} \circ \mathcal{T}$
Capacity Region for Hadamard Channels

Degradable Channels

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- Degrading map $\mathcal{T}: \mathcal{N}^c = \mathcal{T} \circ \mathcal{N}$
- Hadamard channels are degradable
Degradable channel: 

*Bob can simulate Eve*

Degraded map $\mathcal{T}: \mathcal{N}^c = \mathcal{T} \circ \mathcal{N}$

Hadamard channels are degradable

Structure of degrading map:

$$\mathcal{T} = \mathcal{T}_2 \circ \mathcal{T}_1$$

- $\mathcal{T}_1$: von Neumann measurement
- $\mathcal{T}_2$: conditional state preparation
- Gives a single-letter capacity formula
Capacity Region for Hadamard Channels

Single-letterization

Input state $\rho$:

Classical-Quantum trade-off curve:

$$f_\lambda(N) = \max_\rho I(X;B)_N(\rho) + \lambda I(A\rangle BX)_N(\rho)$$

Entanglement-assisted Classical trade-off curve:

$$g_\lambda(N) = \max_\rho I(AX;B)_N(\rho) - \lambda H(A\mid X)_N(\rho)$$

Single-letterization:

$$f_\lambda(N \otimes k) = kf_\lambda(N), \quad g_\lambda(N \otimes k) = kg_\lambda(N)$$

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Capacity Region for Hadamard Channels

Single-letterization

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  \[ f_\lambda(N) = \max_\rho I(X; B)_N(\rho) + \lambda I(ABX)_N(\rho) \]
Capacity Region for Hadamard Channels

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Capacity Region for Hadamard Channels

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- Single-letterization: \( f_\lambda(\mathcal{N} \otimes k) = kf_\lambda(\mathcal{N}) \), \( g_\lambda(\mathcal{N} \otimes k) = kg_\lambda(\mathcal{N}) \)
Trade-off curves for the Dephasing Channel

Figure: Parametrization for qubit $p$-dephasing channel, with $p = 0, 0.1, \ldots, 0.9, 1$: $\mathcal{N} = (1 - p)I + p\Delta$, $I$ identity channel, $\Delta$ Completely dephasing channel
Trade-off curves for $1 \rightarrow N$ Cloning Channels

Figure: Parametrization for $1 \rightarrow N$ Cloning Channels, with $N = 1, 2, 3, 5, 8, 12, 24$
Parametrization of the Triple Trade-Off Region

Figure: Unruh channel with acceleration parameter $z = 0.95$: $z = 0$ identity channel, $z \rightarrow 1$ infinite acceleration
Measuring Improvement over Time-Sharing

- Want a measure of relative improvement
- Ratio of Area under curves

Figure: Relative improvement for (a) classical-quantum and (b) entanglement-assisted classical trade-off
Summary

- Single-letterization of the whole triple trade-off region for Hadamard channels
- Optimal coding strategy for Hadamard channels completely understood
- Parametrization of whole 3D region for 3 natural subclasses of Hadamard channels
- Introduction of measure of relative gain
- Important to consider optimal coding strategy to use resources to full potential